Kawamata-Viehweg Vanishing

Thm (KV I) $n = \dim X$

X smooth projective, L big + nef

Cartier divisor

$$H^i(X, \mathcal{O}_X(K_X + L)) = 0 \quad \text{for } i > 0$$

$$\Updownarrow \text{ SD}$$

$$H^i(X, \mathcal{O}_X(-L)) = 0 \quad \text{for } i < n$$

Thm (KV II = log Kodaira vanishing)

let (X, Δ) snc + projective (klt)

$$\Delta = \sum a_i D_i \qquad 0 \leq a_i < 1$$

L cartier divisor

$$L \equiv M + \Delta \qquad M \text{ ample } \mathbb{Q}\text{-Cartier } \mathbb{Q}\text{-divisor}$$

$$H^i(X, \mathcal{O}_X(K_X + \underbrace{\Delta + M}_{L})) = 0 \quad \text{for } i > 0$$

$$H^i(X, \mathcal{O}_X(-L)) = 0 \quad \text{for } i < n$$

Rmk in KV II, if $a_i \leq 1$

$\Delta' = (1-\varepsilon)\Delta$ coeff < 1 $\quad M' = M + \varepsilon\Delta$ ample

$\Rightarrow H^i(X, \mathcal{O}_X(K_X + M + \Delta)) = H^i(X, \mathcal{O}_X(K_X + M' + \Delta')) = 0$

Prop let (X, Δ) snc pair w/ boundary Δ, $\lfloor \Delta \rfloor = 0$, L cartier divisor with $L \equiv M + \Delta$, then there exists a finite flat $p : Z \to X$ s.t.

$\Delta = \sum a_i D_i$

1) Z smooth

2) $M_Z = p^* M$ cartier

3) $H^i(X, \mathcal{O}_X(-L)) \underset{\text{summand}}{\subseteq} H^i(Z, \mathcal{O}_Z(-M_Z))$

Thm (KV III = log KV I)

let (X, Δ) projective snc pair with $0 \leq a_i < 1$, L cartier divisor with $L \equiv M + \Delta$ for M big + nef $\mathbb{Q}$-divisor

Then $H^i(X, \mathcal{O}_X(K_X + L)) = 0$ for $i > 0$

$$\Updownarrow \quad K_X + \Delta + M$$

$$H^i(X, \mathcal{O}_X(-L)) = 0 \quad \text{for } i < n$$

Thm (KV IV)

let (X, Δ) projective klt pair

L cartier divisor, $L \equiv M + \Delta$ where

M is a big + nef $\mathbb{Q}$-cartier $\mathbb{Q}$-divisor

Then $H^i(X, \mathcal{O}_X(K_X + L)) = 0$ for $i > 0$

$$\Updownarrow$$

$$H^i(X, \mathcal{O}_X(-L)) = 0 \quad \text{for } i < n$$

Outline

$$\text{Kodaira} \xrightarrow{\text{Prop}} \text{KV II} \xrightarrow{\text{log resolution}} \text{KV I}$$

Grauert-Riemenschneider Vanishing

$$\text{KV I} \xrightarrow{\text{Prop}} \text{KV III} \xrightarrow{\text{log resolution}} \text{KV IV}$$

+ stronger GR

Proof of proposition

$$\Delta = \sum_{i=1}^{r} a_i D_i \qquad a_1 = \frac{b}{m} \qquad 0 \le b \le m-1$$

$$\frac{b}{m} D_1$$

Step 1 $\exists \quad f : X' \to X$ finite flat

s.t. 1) $f^* D_1 = mD$

2) X' smooth & $f^*\Delta$ is snc

if D_1 very ample on X

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f"] & \mathbb{P}^N & {[x_0, \dots, x_N]} \arrow[d, mapsto] \\ X \arrow[r] & \mathbb{P}^N & {[x_0^m, \dots, x_N^m]} \end{tikzcd}$$

X' satisfies 2) by Kleiman-Bertini

$f^* D_1 = mD$
where D = hyperplane section of X'

for arbitrary D_1,
write $D_1 = H - H'$ $\quad H, H'$ very ample

Step 2 $f^* \frac{b}{m} D_1 = bD$

$$H^i(X, \mathcal{O}_X(-L)) \underset{\text{summand}}{\subseteq} H^i(X', \mathcal{O}_{X'}(-f^*L))$$

$$\begin{tikzcd} \mathcal{O}_X \arrow[r] & f_* \mathcal{O}_{X'} \arrow[l, bend right, "\frac{1}{\deg f} Tr"'] \end{tikzcd}$$ splits

finite algebra ext in char 0

$$f_* \mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathcal{B}$$

For any $\mathcal{F}$ on X — f finite

$$f_* f^* \mathcal{F} = \mathcal{F} \oplus \mathcal{B} \otimes \mathcal{F}$$

$$H^i(X, \mathcal{F}) \underset{\text{summand}}{\subseteq} H^i(f_* f^* \mathcal{F}) = H^i(X', f^* \mathcal{F})$$

<u>Step 3</u> Use cyclic covering to subtract bD

$f^* D_1 \in |mD| \rightsquigarrow \exists$ cyclic cover

$$g : X'' = X'_{m, f^* D_1} \to X'$$

$$g_* \mathcal{O}_{X''} = \bigoplus_{i=0}^{m-1} \mathcal{O}_{X'}(-iD)$$

+ projection formula

$$\underline{H^i(X'', g^*\mathcal{O}_{X'}(bD - f^*L))} = \bigoplus_{i=0}^{m-1} H^i(X', \mathcal{O}_{X'}((b-i)D - f^*L))$$

$$\text{because } 0 \le b \le m-1 \Rightarrow \quad \begin{array}{c} \cup\!\!\! \mid \\ H^i(X', \mathcal{O}_X(-f^*L)) \\ \cup\!\!\! \mid \ \text{summand} \\ H^i(X, \mathcal{O}_X(-L)) \end{array}$$

$$f \circ g : X'' \to X$$

$$g^*(f^*L - bD) = g^*f^*M + \sum_{i \ge 1} a_i \, g^*f^*D_i$$

$$L \equiv M + \Delta$$

so on X'', $\Delta'' = \sum_{i=2}^{r} a_i \, g^*f^*D_i$ snc

$$0 \le a_i < 1$$

$$L'' = g^*(f^*L - bD)$$

by induction $\Rightarrow$ $h : Z \to X''$

s.t. $H^i(X, \mathcal{O}_X(-L)) \subseteq H^i(X'', \mathcal{O}_X(-L''))$

$$\underset{\text{summand}}{\subseteq} H^i(Z, \mathcal{O}_Z(-h^*g^*f^*M))$$

$$P = f \circ g \circ h : Z \longrightarrow X$$

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Proof of KV

KV II : (X,Δ) as before

$L \equiv M + \Delta$ $\quad$ M ample

by prop, $p : Z \to X$ with

$$H^i(X, \mathcal{O}_X(-L)) \subseteq H^i(Z, \mathcal{O}_Z(-p^*M))$$

p finite $\Rightarrow$ p^*M ample

$\Rightarrow H^i(Z, \mathcal{O}_Z(-p^*M)) = 0 \quad i < n$

by Kodaira

KV I $\quad X$ smooth proj $\quad L$ big+nef Cartier

Ex $\quad L$ big+nef $\iff$ $\exists$ effective N

s.t. $L - \frac{1}{k}N = A_k$

ample for all

$k \gg 0$

take log resolution of (X, N)

$\mu : Y \to X$ $\qquad$ $\mathrm{Ex}(\mu) = \sum E_i$

$L \sim_{\mathbb{Q}} A + \frac{1}{k} N$

$$\mu^* L \sim_{\mathbb{Q}} \mu^* A + \frac{1}{k} \mu_*^{-1} N + \frac{1}{k} \sum a_i E_i$$

pick $k \gg 0$ s.t. $\frac{a_i}{k} < 1$ for all i

Negativity Lemma

$\mu^* A - \sum b_i E_i$ is ample for $0 < b_i \ll 1$

$$\mu^* L \sim_{\mathbb{Q}} \underbrace{(\mu^* A - \sum b_i E_i)}_{A_Y} + \underbrace{\frac{1}{k} \mu_*^{-1} N + \sum \left(\frac{a_i}{k} - b_i\right) E_i}_{\Delta_Y}$$

Now by KV II

$$H^i(Y, \omega_Y \otimes \mu^* \mathcal{O}_X(L)) = 0 \quad \text{for } i > 0$$

Lemma $\mu : Y \to X$ morphism between projective varieties. TFAE

1) $H^j(Y, \mathcal{F} \otimes \mu^* \mathcal{O}_X(H)) = 0$ for H sufficiently ample $j > 0$

2) $R^j \mu_* \mathcal{F} = 0$

PF Leray ss + projection formula

$$H^a(X, R^b\mu_*\mathcal{F} \otimes \mathcal{O}_X(H)) \Rightarrow H^{a+b}(Y, \mathcal{F}(\mu^* H))$$

for H sufficiently ample

$= 0$ for all $a > 0$ by Serre

$\Longrightarrow$ Leray degenerates to

$$H^0(X, R^j\mu_*\mathcal{F} \otimes \mathcal{O}_X(H)) = H^j(Y, \mathcal{F}(\mu^* H))$$

so RHS $= 0$ $\Longleftrightarrow$ $R^j\mu_*\mathcal{F} = 0$

for H sufficiently ample

Want $H^j(X, \omega_X \otimes \mathcal{O}_X(L))$ $\quad j > 0$

Know $H^j(Y, \omega_Y \otimes \mu^* \mathcal{O}_X(L)) = 0$

add rH to L for $r \gg 0$

H ample $\quad j > 0$

as before $\Longrightarrow$ $H^j(Y, \omega_Y \otimes \mu^* \mathcal{O}_X(L + rH)) = 0$

$$\mu^* L = A_Y + \Delta_Y$$

$$\mu^*(L + rH) = \underbrace{A_Y + r\mu^* H}_{\text{ample}} + \Delta_Y$$

but $L + rH$ for $r \gg 0$ is sufficiently ample on X

$\underset{\text{Lemma}}{\Longrightarrow}$ $\boxed{R^j \mu_* \omega_Y = 0 \quad (\text{GR vanishing}) \quad j > 0}$

$\mu_* \omega_Y = \omega_X$ $\quad$ X is smooth so in particular canonical

$$H^i(X, R^j \mu_* \omega_Y \otimes \mathcal{O}_X(L)) \Rightarrow H^{i+j}(Y, \omega_Y \otimes \mathcal{O}_Y(\mu^* L))$$

ss degenerates to

$$H^i(X, \underset{\underset{0}{\shortparallel}}{\mu_* \overset{\overset{\omega_X \otimes \mathcal{O}_X(L)}{\shortparallel}}{\omega_Y}} \otimes \mathcal{O}_X(L)) = H^i(\underset{\underset{0 \quad i > 0}{\shortparallel}}{Y, \omega_Y \otimes \mathcal{O}_Y(\mu^* L)})$$

<u>KV III</u> $\quad (X, \Delta)$ snc, $\quad \Delta$ boundary $\lfloor \Delta \rfloor = 0$

$L \equiv M + \Delta$ $\quad$ M big + nef

by Prop $\quad$ $p : Z \to X$ flat finite

Z smooth

$$H^i(X, \mathcal{O}_X(-L)) \underset{\text{summand}}{\subseteq} H^i(Z, \mathcal{O}_Z(-p^*M))$$

$\|$ $\quad$ p^*M big+nef

0 $\quad$ for $i < n$