Kawamata-Viehweg vanishing

Thm (KV III) X is smooth projective with snc $\Delta = \sum a_i D_i$, $0 \le a_i < 1$, $a_i \in \mathbb{Q}$

L Cartier divisor s.t.

$L \equiv M + \Delta$ where M is a big + nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor

Then $H^i(X, \mathcal{O}_X(K_X + L)) = 0$ for $i > 0$

$\Updownarrow$

$H^i(X, \mathcal{O}_X(-L)) = 0$ for $i < \dim X$

Thm (KV III') let X be a smooth proj variety & $\Delta = \sum d_i D_i$,

$D := L + \Delta$ L cartier

D big + nef $\lceil \Delta \rceil - \Delta$ has snc support

Then $H^i(X, \mathcal{O}_X(K_X + \lceil D \rceil)) = 0$

$\Delta' = \lceil \Delta \rceil - \Delta$ $\lceil D \rceil = D + \Delta'$ So this is KV III to (X, Δ') & $D + \Delta'$, $M = D$

$\lfloor \Delta' \rfloor = 0$ D big + nef

GR

Cor (generalized Grauert-Riemenschneider Vanishing)

$f: Y \to X$ is a birational morphism between projective varieties & suppose (Y, Δ) is snc, $\lfloor \Delta \rfloor = 0$, Δ effective

L is a cartier divisor

$L \equiv M + \Delta$ where M is a nef $\mathbb{Q}$-divisor

Then $R^i f_* \mathcal{O}_Y(K_Y + L) = 0$ for $i > 0$

Proof fix H ample on X

our lemma from last time told us that

$$R^i f_* \mathcal{O}_Y(K_Y + L) = 0$$

$$\Longleftrightarrow$$

$$H^i(Y, \mathcal{O}_Y(K_Y + L) \otimes f^* \mathcal{O}_X(rH)) = 0$$

for all $r \gg 0$

$$L + f^* rH \equiv \underbrace{M + f^* rH}_{\text{big + nef}} + \Delta$$

follows from KV III

Rmk Often applied to say that $R^i f_* \omega_Y = 0$, $i > 0$, for $f: Y \to X$ birational with Y smooth

Rmk Can extend this and relative KV vanishing to the case where f is projective by compactifying

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Thm (KV IV) let (X, Δ) be a projective klt pair, L be a ($\mathbb{Q}$-)Cartier divisor s.t. $L \equiv M + \Delta$ with M a big + nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor

Then

$$H^i(X, \mathcal{O}_X(K_X + L)) = 0 \quad \text{for} \quad i > 0$$

Pf let $f: Y \to X$ be a log resolution

$$K_Y + f_*^{-1}\Delta + f^*M = f^*(K_X + \Delta) + \overbrace{\sum a_i E_i}^{E_Y} + f^*M$$

$a_i > -1$

$$C_Y = \sum_{c_i < 1} c_i E_i \qquad \lceil E_Y \rceil \geq 0$$

$$K_Y + f_*^{-1}\Delta + C_Y + f^*M = f^*(K_X + \Delta) + f^*M + \lceil E_Y \rceil$$

effective simple normal crossings

$\lfloor C_Y \rfloor = 0$

effective exc

f^*M is big + nef, $\Delta_Y = f_*^{-1}\Delta + C_Y$

Now (Y, Δ_Y) is an snc pair with Δ_Y effective $\lfloor \Delta_Y \rfloor = 0$

$$\Rightarrow H^i(Y, \mathcal{O}_Y(K_Y + \Delta_Y + f^*M)) = H^i(Y, f^*\mathcal{O}_X(K_X+\Delta+M)(\lceil E_Y \rceil))$$

$i>0$ $\quad \| \quad$ by KV III

0

$$f_*(f^*\mathcal{O}_X(K_X+\Delta+M)(\lceil E_Y \rceil)) =$$

by same argument as in the equivalent char of canonical sing

$$f_* f^*\mathcal{O}_X(K_X+\Delta+M)$$

$$= \mathcal{O}_X(K_X+\Delta+M)$$

projection formula + $f_*\mathcal{O}_Y = \mathcal{O}_X$

By GR vanishing, $R^j f_* \mathcal{O}_Y(f^*(K_X+\Delta+M) + \lceil E_Y \rceil)$

$$= R^j f_* \mathcal{O}_Y(K_Y + \Delta_Y + f^*M)$$

$$= 0$$

Leray $\Rightarrow$ $H^i(X, \mathcal{O}_X(K_X+\Delta+M)) = H^i(Y, f^*\mathcal{O}_X(K_X+M+\Delta)(\lceil E_Y \rceil))$

$= 0$

$\boxtimes$

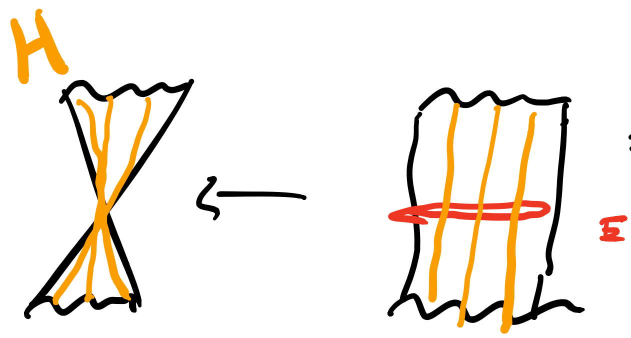
Cor (relative KV) $f : (X, \Delta) \to Z$ morphism of projective varieties with (X, Δ) klt, Δ effective, L a $\mathbb{Q}$-Cartier divisor with $L \equiv M + \Delta$ where

M is f-big + f-nef

Then $R^i f_* \mathcal{O}_X(K_X + L) = 0$ for $i > 0$

proof exercise

Ex (Failure of KV vanishing when $\lfloor \Delta \rfloor \neq 0$)



= Total space of $\mathcal{O}_{\mathbb{P}^2}(-1)|_E$ = X

cone over an elliptic curve $\subseteq \mathbb{P}^2$

(X, E) log canonical snc pair

$M = f^* H$ big + nef

$L = M + E$ $\qquad$ $\Delta = E$

$$0 \to \mathcal{O}_X(M) \to \mathcal{O}_X(L) \to \mathcal{O}_E(L|_E) \to 0 \qquad \text{\& KV}$$

$$0 = H^1(\mathcal{O}_X(M)) \hookrightarrow H^1(\mathcal{O}_X(L)) \to H^1(\mathcal{O}_E(L|_E)) \twoheadrightarrow H^2(\mathcal{O}_X(M)) = 0$$

$$L|_E = (f^*H + E)|_E = \text{degree } -3$$

$$\Longrightarrow \boxed{H^1(\mathcal{O}_E(L|_E)) \neq 0 \,!}$$

The issue is that M fails to be big on the non-klt center! (lc centers)

Thm (log canonical KV)

$f : (X, \Delta) \to Z$ projective morphism w/ X smooth, Δ snc boundary, let L Cartier

$L \equiv M + \Delta$ s.t.

1) M f-nef + f-big
2) M restricted to each component of $\mathrm{Nklt}(X, \Delta)$ is f-big

Then $R^i f_* \mathcal{O}_X(K_X + L) = 0$ for $i > 0$.

Proof:

Induct on dimension: if $n = \dim X = 1$ then theorem is true

Suppose $n > 1$

$\lfloor \Delta \rfloor = \sum_{i=1}^{r} D_i$ we will induct on r

$\underline{r=0}$ $\lfloor \Delta \rfloor = 0$ so we get KV III

$\underline{r>0}$ $D_1 \in \lfloor \Delta \rfloor$

$$0 \to \mathcal{O}_X(K_X + L - D_1) \to \mathcal{O}_X(K_X+L) \to \mathcal{O}_{D_1}((K_X+L)|_{D_1}) \to 0$$

$$L - D_1 = M + \sum_{i=2}^{s} a_i D_i$$

$$(K_X + D_1 + L - D_1)|_{D_1} = K_{D_1} + \underbrace{(L-D_1)|_{D_1}}_{\text{snc}}$$

D_1 smooth

$R^i f_* \mathcal{O}_X(K_X + L - D_1) = 0$ $i>0$

by induction on r

by induction on n,

$$R^i f_* \mathcal{O}_{D_1}(K_{D_1} + (L-D_1)|_{D_1}) = 0 \quad i>0$$

$$(L-D_1)|_{D_1} = M|_{D_1} + \sum_{i=2}^{s} a_i D_i|_{D_1}$$

f-big + f-nef by assumption

$$\Longrightarrow \quad R^i f_* \mathcal{O}_X(K_X+L) = 0$$

for $i>0$

Cone theorems

(Mori, Kawamata, Reid, Shokurov, Kollár 80's-90's)

Thm (Base point free theorem)

Let (X, Δ) be a projective klt pair with Δ effective. D is a nef Cartier divisor s.t.

$$\underbrace{aD - (K_X + \Delta)}_{M} \text{ is big + nef for some } a > 0$$

Then D is semi-ample. $|bD|$ is bpf for $b \gg 0$

if $K_X + \Delta + M$ is nef + M is big + nef $\Rightarrow$ $K_X + \Delta + M$ is semi-ample

Thm (Non-Vanishing theorem)

Let X projective, D a nef Cartier divisor, G a $\mathbb{Q}$-divisor

1) $aD - (K_X - G)$ is a big + nef $\mathbb{Q}$ Cartier div for some $a > 0$

2) $(X, -G)$ is klt

Then $H^0(X, \mathcal{O}_X(mD + \lceil G \rceil)) \neq 0$ for all $m \gg 0$

Rationality theorem

let (X, Δ) projective klt, Δ effective

fix $m > 0$ s.t. $m(K_X + \Delta)$ is Cartier

suppose $K_X + \Delta$ not nef

fix H a big + nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor

$$r = r(H) := \sup\{ t \in \mathbb{R} \mid H + t(K_X + \Delta) \text{ is nef} \}$$

then $r = \frac{u}{v} \in \mathbb{Q}$ s.t.

$$0 < v \leq m(\dim X + 1)$$

Thm (Cone + Contraction)

let (X, Δ) projective klt, $\Delta \geq 0$

1) there exist countably many C_i rational s.t.

$$0 < -(K_X + \Delta) . C_i \leq 2 \dim X$$

& $\overline{NE}(X) = \overline{NE}(X)_{(K_X + \Delta) \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$

2) for any ample H $\varepsilon > 0$

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X + \Delta + \varepsilon H) \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i]$$

3) $F \subseteq \overline{NE}(X)$ extremal face

which is $(K_X + \Delta) < 0$

then $\exists !$ projective

$\varphi_F : \mathrm{Cont}_F : X \to Z$ s.t.

i) $\varphi_{F*} \mathcal{O}_X = \mathcal{O}_Z$

ii) $\varphi_F(C) = pt \iff [C] \in F$

4) for any Cartier L s.t.

$L.C = 0$ for all $[C] \in F$

there exists Cartier L_Z

s.t. $\varphi_F^* L_Z = L$