Thm (Non-Vanishing theorem)

X projective, D a nef Cartier divisor,

G a $\mathbb{Q}$ divisor such that

1) $aD + G - K_X$ is big + nef for some $a > 0$

$\mathbb{Q}$-Cartier

2) $(X, -G)$ is klt

Then for $m >> 0$

$$H^0(X, mD + \lceil G \rceil) \neq 0$$

Thm (base point free theorem)

X projective, (X, Δ) klt pair with Δ effective. D nef Cartier divisor such that

$$aD - (K_X + \Delta) \text{ is big + nef}$$

for some $a > 0$

then D is semiample

$|bD|$ is bpf for $b >> 0$

Non-Vanishing ⇒ bpf ⇒ Rationality ⇒ cone

bpf + cone ⇒ contraction

Strategy (bpf)

X smooth, $\Delta = 0$, M is ample

F effective integral

$$0 \to \underline{\mathcal{O}_X(K_X+M)} \to \mathcal{O}_X(K_X+F+M) \to \mathcal{O}_F(K_F+M|_F) \to 0$$

$$(K_X+F)|_F = K_F$$

By Kodaira vanishing $H^1(X, \mathcal{O}_X(K_X+M)) = 0$

$$H^0(X, \mathcal{O}_X(K_X+F+M)) \twoheadrightarrow H^0(F, \mathcal{O}_F(K_F+M|_F))$$

$$bD - K_X \overset{\uparrow}{=} M + F$$

if we can write

Need Kawamata-Viehweg vanishing holds for K_X+M

Obs 1

$f: Y \to X$ proper birational map

$bf^*D + E$

E effective + f-exceptional

$$H^0(X, \mathcal{O}_X(bD)) = H^0(Y, \mathcal{O}_Y(bf^*D))$$
$$= H^0(Y, \mathcal{O}_Y(bf^*D + E))$$

Want

$$bf^*D - K_Y = \underbrace{N}_{\text{big + nef}} + \underbrace{\Delta_Y}_{\text{klt boundary}} + \underbrace{F}_{\text{irreducible}} - \underbrace{E}_{\text{f-exc + effective}}$$

$F \not\subseteq \mathrm{Supp}(\Delta_Y)$ ⇝ $(N + \Delta_Y)|_F$

$$\underbrace{N_F}_{\text{satisfies hypothesis}} + \underbrace{\Delta_F}_{\text{klt for F}}$$

Obs 2

Can assume

$$f^*|mD| = \underbrace{|L|}_{\substack{\text{moving part}\\ \text{+ bpf}}} + \underbrace{\overbrace{\sum r_j E_j}^{\text{f-exc}} + \overbrace{\sum r_k F_k}^{\text{not exc}}}_{\text{fixed}}$$

$r_j, r_k > 0$

$$b f^* D - K_Y = \underbrace{\underbrace{(b - cm - a) f^* D}_{\text{nef}} + \underbrace{cL}_{\text{bpf}} + \underbrace{f^*(aD - K_X)}_{\text{big + nef}}}_{= N(b,c) \text{ big + nef}} - \sum a_j E_j + c \sum r_j E_j + c \sum r_k F_k$$

if $c \geq 0$

$b \geq cm + a$

$$K_Y = f^* K_X + \sum a_j E_j$$

$$\overbrace{\sum (c r_j - a_j) E_j + \sum c r_k F_k}^{B(c)}$$

$$= \Delta_Y + F - E$$

pick c s.t.

$$\max\{ c r_j - a_j , c r_k \} = 1$$

$$\min\{ -(\quad), - \quad \} = -1$$

$F := \lfloor B(c) \rfloor$ reduced integral divisor

$$B(c) - \lfloor B(c) \rfloor =: \Delta_Y - E$$

effective

Need Δ_Y to be a klt boundary

E effective + exceptional

since $\lfloor \beta(c) - \lfloor B(c) \rfloor \rfloor \leq 0$

$\Rightarrow \Delta_Y$ klt boundary

$-E = \sum (cr_j - a_j) E_j$ is exceptional

$cr_j - a_j \leq 0$

So we've written

$$bf^*D - K_Y = \underset{\text{big + nef}}{N} + \underset{\text{klt boundary}}{\Delta_Y} + \overset{\text{integral + } F \not\subseteq \mathrm{Supp}(\Delta_Y)}{F} - \overset{\text{effective exceptional}}{E}$$

issues 1) F is not irreducible

2) $N|_F$ doesn't have to be big + nef

Solution we perturb by

$$\sum p_j F_j$$

to make F irreducible & N ample

$0 < p_j \ll 1$

Proof that nonvanishing $\Rightarrow$ bpF

D nef, (X, Δ) proj klt

$aD - (K_X + \Delta)$ is big+nef for some $a > 0$

Step 1 $|mD| \neq \emptyset$ for $m \gg 0$

$f : Y \to X$ log resolution s.t.

1) $K_Y = f^*(K_X + \Delta) + \sum a_j F_j \qquad a_j > -1$

2) $\underline{f^*(aD - (K_X + \Delta)) - \sum p_j F_j}$ is ample for $0 < p_j \ll 1$

($aD - (K_X + \Delta)$ big+nef $\Longleftrightarrow$ effective

$aD - (K_X + \Delta) = \underset{\text{ample}}{A_k} + \frac{1}{k} N$)

$$a f^* D - K_Y + \underbrace{\sum (a_j - p_j) F_j}_{G}$$

$a_j - p_j > -1 \implies \lceil G \rceil$ effective

Claim $\lceil G \rceil$ is f-exceptional

$a_j - p_j > 0$ $\quad F_j \subseteq \lceil G \rceil$

iff $a_j > p_j > 0$

$\implies F_j$ exceptional b/c

Δ is effective

$$H^0(Y, \mathcal{O}_Y(mf^*D + \lceil G \rceil)) = H^0(X, \mathcal{O}_X(mD))$$

$\neq$

0 by nonvanishing b/c

$\Uparrow$

$af^*D + C - K_Y$ is big + nef

Step 2 since $|mD| \neq \emptyset$ for $m \gg 0$

stable base locus

Noetherian induction

$$\bigcap_{m \in \mathbb{N}} Bs(|mD|) = \mathbb{B}(D) = Bs(|mD|) \text{ for some } m \gg 0$$

set theoretic

Fix such an m

Suppose $B_s(|mD|) \neq \emptyset$

pick log resolution satisfying

1) + 2) from step 1 as

well as

3) $f^*|mD| = \underset{\substack{\uparrow \\ \text{bpf}}}{|L|} + \underset{\substack{\uparrow \\ \text{fixed}}}{\sum r_j F_j}$ $\quad r_j \geq 0$

a) $f^{-1}(B_s(|mD|)) = Bs(f^*|mD|)$ as sets

b) $B_s(|mD|) = \bigcup\{f(F_j) \mid r_j > 0\}$

proof by contradiction:

find an F_k w/ $r_k > 0$

s.t. $F_k \not\subseteq B_s(|mD|)$

Step 3 apply the strategy

$$N(b,c) = bf^*D - K_Y + \sum(a_j - cr_j - p_j)F_j$$

$$\equiv \underbrace{(b - cm - a) f^* D}_{\text{big + neg}}$$

$$+ \underbrace{cL}_{\text{bpf}}$$

$$+ \underbrace{f^*(aD - (K_X + \Delta)) - \sum p_j F_j}_{\text{ample}}$$

as long as

$e > 0$ $\quad b \geq cm + a$

$$\lceil N(b, c) \rceil = a f^* D - K_Y + \underbrace{\sum \lceil a_j - c r_j - p_j \rceil F_j}_{\lceil A \rceil \overset{\shortparallel}{-} F}$$

Pick p_j & c s.t.

$\min \{ a_j - c r_j - p_j \} = -1$ + achieved for a unique $j = k$

$$\sum (a_j - c r_j - p_j) F_j = \underset{n}{A} - F$$

$F = F_k$ $\qquad \underset{\text{eff exc}}{E} - \underset{\text{rel boundary}}{\Delta_Y}$

Step 4 lifting sections

$$K_Y + \lceil N(b, c) \rceil \equiv b f^* D + \lceil A \rceil - F$$

$$0 \to \underset{\underset{K_Y + \lceil \text{ample} \rceil}{\cong}}{\mathcal{O}_Y(bf^*D + \lceil A \rceil - F)} \to \mathcal{O}_Y(bf^*D + \lceil A \rceil) \to \mathcal{O}_F((bf^*D + \lceil A \rceil)|_F) \to 0$$

$H^1(\mathcal{O}_Y(K_Y + \lceil \text{ample} \rceil)) = 0$

So $H^0(Y, \mathcal{O}_Y(bf^*D + \lceil A \rceil)) \twoheadrightarrow \overset{\neq 0}{H^0(F, \mathcal{O}_F(bf^*D + \lceil A \rceil))}$

$$(bf^*D + \lceil A \rceil)|_F - K_F = (bf^*D + \lceil A \rceil - K_X - F)|_F$$

$$= N(b, c)|_F$$

is ample

$6 = \lceil A \rceil|_F \Longrightarrow$ nonvanishing on F

$\Longrightarrow$

$$0 \neq s \in H^0(\mathcal{O}_Y(bf^*D + \lceil A \rceil)) = H^0(\mathcal{O}_X(bD))$$

$$\downarrow$$

$s|_F \neq 0$

$F = F_k$ $\quad \lceil A \rceil$ effective exceptional

$\Longrightarrow f(F_k) \not\subseteq B_s(|bD|)$ for all large $b \gg 0$

$\boxtimes$