Thm (Base point free theorem)

$(X,\Delta)$ projective klt, $D$ nef Cartier

$aD-(K_X+\Delta)$ is big + nef for some $a>0$

Then $D$ is semiample.

Thm (Non vanishing theorem)

$(X,-G)$ Projective klt, $D$ nef

$-G$ not necesarily a boundary

s.t. $aD-(K_X-G)$ is big + nef

$\Rightarrow H^0(X, mD+\lceil G\rceil)\neq 0$ for $m>0$

Rmk: $\lceil G\rceil$ effective, & if $-G$ is a boundary, $\Rightarrow \lceil G\rceil=0$

Last time

non vanishing $\Rightarrow$ basepoint free

Strategy

$(X,\Delta)$ take $f:Y\to X$ a log resolution

Want

$$bf^*D-K_Y=\underset{\text{ample}}{N(b,c)}+\underset{\text{smooth irreducible reduced}}{F}+\underbrace{\Delta_Y-E}_{-A}$$

klt boundary

effective exceptional

split $bD$ into its moving + fixed parts, add discrepancies
perturb by a small generic f-ample

$$K_Y + F + \lceil N(b,c) \rceil = bf^*D + \lceil A \rceil$$

If $F$ shares no components with $N(b,c)$
then restriction to $F$ ample

$$K_F + \lceil N(b,c) \rceil|_F = (bf^*D + \lceil A \rceil)|_F$$

1) left hand side has sections by induction

2) KV $\Rightarrow$ $H^1(Y, K_Y + \lceil N(b,c) \rceil) = 0$
$\Rightarrow$ restriction map is surjective
so $bf^*D + \lceil A \rceil$ has sections on $Y$

3) We arrange it so that $\lceil A \rceil$ is effective $f$-exceptional $+f^*\lceil G \rceil$

$$\Rightarrow H^0(Y, bf^*D + \lceil A \rceil) = H^0(X, bD + \lceil G \rceil)$$

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## Proof of non-vanishing

**Step 1** without loss of generality, we can assume $aD - (K_X - G)$ is ample & $(X, -G)$ is snc let $f: X' \to X$ be a log resolution

$f^*(K_X - G) = K_{X'} - G'$ $\quad f_* G' = G$

the disc of $p$ over $(X, -G)$ are the same as those $(X', -G')$

$\Rightarrow (X', -G')$ klt

$$a f^* D - (K_{X'} - G') = f^*(aD - K_X + G)$$

big + nef

Pick some small $f$-exceptional effective

$$E = \sum p_j E_j \qquad 0 < p_j \ll 1$$

s.t. $f^*(aD - K_X + G) - E$ is ample

$G'' = G' - E$

$$\quad = a f^* D - K_{X'} + G''$$

$G' = G'' +$ $f$-exceptionals

$$\underline{H^0(X', mf^*D + \lceil G'' \rceil)} \subseteq H^0(X', mf^*D + \lceil G' \rceil)$$

$$= H^0(X, mD + \lceil G \rceil)$$

**Step 2** suppose $D \equiv 0$

$G \equiv K_X + A$ where $A$ is ample

$$h^0(X, mD + \lceil G \rceil) \underset{\text{by KV vanishing}}{=} \chi(mD + \lceil G \rceil) = \chi(\lceil G \rceil) \underset{\text{RR}}{=} h^0(X, \lceil G \rceil) \underset{\text{KV}}{\neq} 0$$

$(X, -G)$ klt $\Rightarrow$ $\lceil G \rceil \geq 0$

so wlog assume $D \not\equiv 0$

$\Rightarrow$ $\exists$ a curve $C \subseteq X$ s.t. $D.C > 0$

**Step 3** pick some $x \in X$ away from $G$

**Claim** $\forall\, q \gg 0$ $\exists$ $M(q) \in |qD - K_X + G|$

with $\mathrm{mult}_x M(q) > 2 \underbrace{\dim X}_{d}$

since $D$ is Nef $\Rightarrow$ $D^e . A^{d-e} \geq 0$ for $e \geq 0$ & $A$ ample

for $q \gg 0$

$$\underbrace{(qD - K_X + G)^d}_{N(q)} = [\underbrace{(q-a)D}_{\text{nef}} + \underbrace{(aD - K_X + G)}_{\text{ample}}]^d$$

$$\Longrightarrow \quad \geq d(q-a)D \cdot \underbrace{(aD - K + G)^{d-1}}_{} > 0$$

Since $\exists$ C s.t. $D.C > 0$

& since ample

$$\Longrightarrow (aD - K + G)^{d-1} \equiv C' + E$$

s.t. $D.C' > 0$, E effective

in fact, as $q \to \infty$,

$$(qD - K_X + G)^d \longrightarrow \infty$$

$$h^0(t(qD - K + G)) \geq \frac{t^\delta}{d!} N(q) + \text{lower order terms}$$

Codimension of the locus of divisors with mult $> 2 \dim X t$

grows as $\dfrac{t^\delta}{d!}(2d)^\delta$ + lower order

So for $q$ & $t$ large, the difference becomes large $\Longrightarrow$

$$M(q,t) \in |t(qD - K + G)|$$

with $\mathrm{mult}_x M(q,t) > 2dt$

$$M(q) := \frac{M(q,t)}{t} \equiv_{\mathbb{Q}} qD - K + G$$

has multiplicity $> 2d$

<u>Step 4</u> pick $f = f(q) : Y \to X$

a log resolution of $(X, G + M(q))$

s.t. $f$ factors through

$Bl_x X \to X$

1) $K_Y = f^*(K_X - G) + \sum b_j F_j \qquad b_j > -1$

2) $\frac{1}{2} f^*(aD + G - K) - \sum p_j F_j$ is ample

for $0 < p_j \ll 1$

3) $f^* M(q) = r_0 F_0 + \sum_{\geq 1} r_j F_j$

$F_0 =$ exceptional of $Bl_x X$ $\qquad r_0 > 2d$

$$K_Y + \Delta_Y + F + N(b,c) = bf^*D + E$$

$$N(b,c) = bf^*D - K_Y + \sum(-cr_j + b_j - p_j)F_j$$

$$= f^*(bD - K_X + G) - cf^*M(q) - \sum p_j F_j$$

$$\equiv f^*(bD - K_X + G - cqD + cK - cG) - \sum p_j F_j$$

$$\equiv f^*([b - cq - a(1-c)]D + (1-c)(aD - K + G)) - \sum p_j F_j$$

$$\equiv f^*(\underbrace{([b - cq - \underbrace{a(1-c)}_{\geq 0})D}_{\text{nef}} + \underbrace{(\tfrac{1}{2} - c)}_{\geq 0}(aD - K + G))}_{\text{big + nef}} + \underbrace{\tfrac{1}{2} f^*(aD - K + G) - \sum p_j F_j}_{\text{ample}}$$

So $N(b, c)$ is ample if

$c \leq 1/2$ $\quad b \geq a + c(q - a)$

<u>Step 5</u> let $c = \min \{ (1 + b_j - p_j)/r_j \} > 0$

Pick $p_j$ generically so that $c$ is achieved exactly once

$F_j = F$

need to check that $c < 1/2$

$b_0 = d - 1 \qquad r_0 > 2d$

$$0 < c \leq \frac{d-\varepsilon}{2d} < 1/2$$ ✓

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$$N(b,c) = bf^*D - K_Y + \sum \underbrace{(-cr_j + b_j - p_j)F_j}_{= A - F}$$

So $\lceil A \rceil$ is effective

$$K_Y + \underline{N(b,c)} + F = bf^*D + A$$

$$f^*G = \sum g_j F_j \quad \text{if } F_j \text{ not exceptional then } g_j = b_j$$

$$(-cr_j + b_j - p_j) < b_j = g_j$$

$$\lceil A \rceil \leq f^*\lceil G \rceil + \text{effective exceptionals}$$

exceptionals don't contribute to $H^0$

$$H^0(Y, bf^*D + \lceil A \rceil)$$
$$\subseteq H^0(Y, bf^*D + f^*\lceil G \rceil)$$
$$= H^0(X, bD + \lceil G \rceil) \neq 0$$

by KV vanishing

$$H^0(Y, bf^*D + \lceil A \rceil) \twoheadrightarrow H^0(F, bf^*D|_F + \lceil A \rceil|_F)$$

$$\left(bf^*D + \lceil A \rceil\right)\big|_F \equiv K_F + N(b,c)\big|_F$$

induction on dimension

ample + snc