Non-vanishing $\Rightarrow$ basepoint free (Last time)

Abundance Conjecture

(X, Δ) projective log canonical, Δ effective

then 1) $$R(K_X + \Delta) := \bigoplus_{m \geq 0} H^0(X, mK_X + \lfloor m\Delta \rfloor)$$

finitely generated

2) if $K_X + \Delta$ is nef

then $K_X + \Delta$ is semiample

Cor let (X, Δ) is a projective klt pair & $K_X + \Delta$ is $\boxed{\text{big}}$ + nef

log general type

then abundance holds.

proof pick $m \geq 1$ s.t.

$D := m(K_X + \Delta)$ is Cartier + nef

$D - (K_X + \Delta)$ is big + nef

so by bpf, D is semiample

$\Longrightarrow \quad b(K_X+\Delta)$ is basepoint free

it defines $f: X \to Z$

$$b(K_X+\Delta) = f^*H$$

$$\mathcal{G}_m = f_* \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor)$$

$$\mathcal{G}_m \otimes \mathcal{O}_Z(H) = \mathcal{G}_{m+b}$$ by projection formula

---

$$R(K_X+\Delta) = \bigoplus_{m \geq 0} H^0(Z, \mathcal{G}_m)$$

$$R(H) = \bigoplus H^0(Z, H)$$

$R(K_X+\Delta)$ is a f.g. module over $R(H)$

but $\underline{R(H)}$ is f.g. b/c H ample.

Thm (Rationality theorem)

Let (X, Δ) be a projective klt pair with Δ effective, $K_X + \Delta$ <u>not</u> nef

fix a s.t. $a(K_X+\Delta)$ is cartier

let H be a big + nef Cartier divisor

$$r := \max\{t \in \mathbb{R} \mid H + t(K_X + \Delta) \text{ is nef}\}$$

then $r = \frac{u}{v} \in \mathbb{Q}$ with
$0 < v < a(\dim X + 1)$

Proof

$$\frac{q-1}{p} < r < \frac{q}{p}$$

$pH + (q-1)(K_X + \Delta)$ big + nef

$pH + q(K_X + \Delta)$ not nef so not semi ample

$H^i(X, K_X + \Delta + D) = 0$ for $i > 0$ by KV vanishing

if r is not rational
$\Longrightarrow$ many such pairs (p, q)
with $pH + q(K_X + \Delta)$ effective

Step 1 wlog H is base point free

$$H' = m(\underbrace{cH + da(K_X + \Delta)})$$

$m \gg c \gg d > 0$

semiample by bpf

$$H + r(K_X + \Delta) \sim_{\mathbb{Q}} H' + r'(K_X + \Delta)$$

$$\Longrightarrow \quad r = \frac{r' + mda}{mc}$$

$r' = \frac{u'}{v'}$ rat'l $\Rightarrow$ $r = \frac{u}{v}$ rat'l + v divides v'

Step 2 Y smooth proj, $\{D_i\}_{i=1}^{k}$ cartier

A snc s.t. $\lceil A \rceil \geq 0$

$(Y, -A)$ klt

$$P(u_1, \ldots, u_k) = \chi\left(\sum u_i D_i + \lceil A \rceil\right)$$

suppose that 1) $\sum u_i D_i$ is nef

2) $\sum u_i D_i - (K_Y - A)$ is ample

then 1) $H^i(m \sum u_i D_i + \lceil A \rceil) = 0$ for $i > 0$ $m > 0$ by KV vanishing

2) $|m \sum u_i D_i + \lceil A \rceil| \neq \emptyset$ for all $m \gg 0$

so $P(u_1, \dots, u_k) \not\equiv 0$ with deg $\leq \dim Y$

<u>Step 3</u> rationality criteria

<u>Lemma</u> Suppose $P(x,y) \in \mathbb{Z}[x,y]$ is not identically zero, of degree $\leq n$

fix $a \in \mathbb{Z}$, $\varepsilon > 0$, $r \in \mathbb{R}$ s.t.

$P(x,y) = 0$ for all sufficiently large integers satisfying

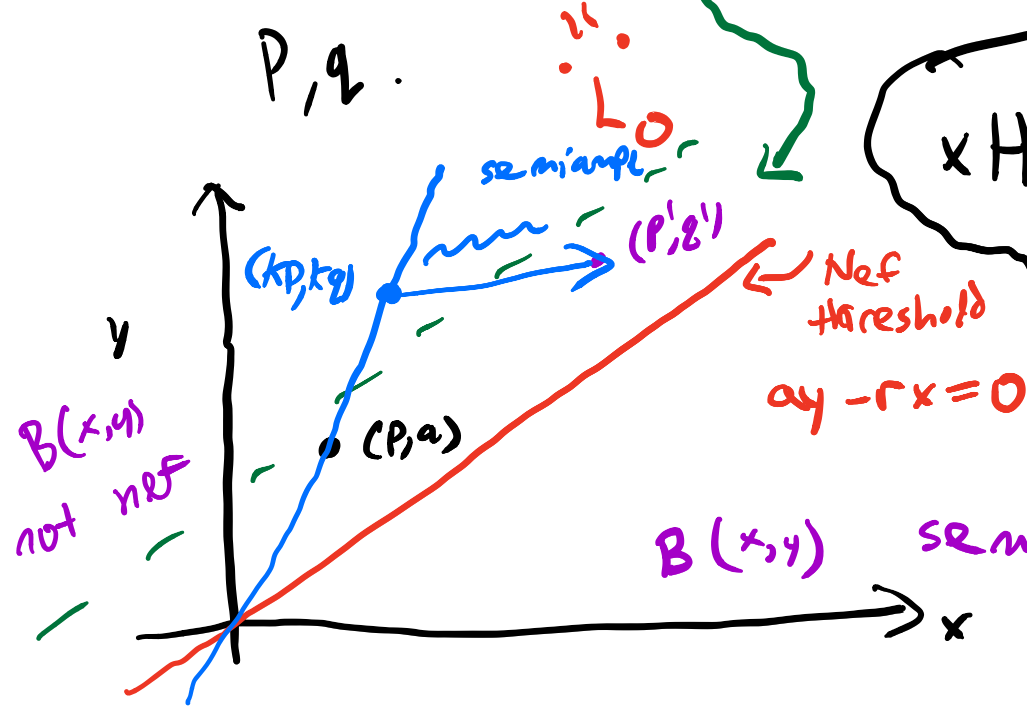
$$0 < ay - rx < \varepsilon$$

$$\Longrightarrow \quad r = \frac{u}{v} \quad \text{with} \quad 0 < v \le \frac{a(n+1)}{\varepsilon}$$

Step 3 let a, r be as in the hypotheses of the the theorem

$$L(p,q) = \mathbf{B}_s|pH + qa(K_X + \Delta)|$$

Claim fix $\varepsilon > 0$, for large enough p, q satisfying $0 < aq - rp < \varepsilon$, then $L(p,q)$ is independent of p, q.



$$xH + ya(K_X + \Delta) =: B(x,y)$$

so for large (p', q') inside this strip, we get $B(p', q') \equiv k\, B(p, q)$ + semiample

$\implies L(p', q') \subseteq L(p, q)$

for $p', q' \gg 0$ inside this strip

by Noetherian, this chain stabilizes to some $L_0 \neq \emptyset$

<u>set</u> $I \subseteq \mathbb{Z} \times \mathbb{Z}$ to be the set of (p, q) inside the strip with $L(p, q) = L_0$

$\varepsilon = 1$

<u>Step 5</u> $g : Y \to X$ log resolution of (X, Δ)

$D_1 = g^* H$, $D_2 = g^* a(K_X + \Delta)$

$A = A_Y(X, \Delta)$

$K_Y = g^*(K_X + \Delta) + A$

klt $\implies \lceil A \rceil$ effective g-exceptional

$$K_Y - A = g^*(K_X + \Delta)$$

$$xD_1 + yD_2 = g^* B(x,y)$$

$$P(x,y) = \chi(xD_1 + yD_2 + \lceil A \rceil) \not\equiv 0$$

by step 2

since $\lceil A \rceil$ is f-exc

$$H^0(Y, xD_1 + yD_2) = H^0(Y, g^* B(x,y) + \lceil A \rceil)$$
$$= H^0(X, B(x,y))$$
$$= H^0(X, xH + ya(K_X + \Delta))$$

Step 6

$$xD_1 + yD_2 - (K_Y - A) = g^*(B(x,y) - (K_X + \Delta))$$
$$= g^*(B(x, y - 1/a))$$

if

$0 < ay - bx < 1$ $\Longrightarrow$ $B(x, y - 1/a)$ big + nef

$xD_1 + yD_2 + A - K_Y$ is big + nef

$\Rightarrow \quad H^i(Y, xD_1 + yD_2 + \lceil A \rceil) = 0$

for $i > 0$ by KV

Suppose that $r \notin \mathbb{Q}$

then by the lemma,

we have infinitely many (p, q)

inside $0 < aq - rp < 1$ s.t.

$$0 \neq P(p, q) = h^0(Y, pD_1 + qD_2 + \lceil A \rceil)$$

$$= h^0(X, \underbrace{pH + qa(K_X + \Delta)}_{B(p,q)})$$

$$L_0 = L(p, q) \neq X$$

for $p, q \gg 0$ in the strip

i.e. for $(p, q) \in I$

Step 7 $f : Y \to X$ be a log resolution such that

1) $K_Y \equiv f^*(K_X + \Delta) + \sum a_j F_j \quad a_j > -1$

2) $f^*(pH + (qa-1)K_X + \Delta) - \sum p_j F_j$

$\quad \overset{\shortparallel}{B(p, q - 1/a)}$ ample for $a < p_j \ll 1$

big + nef

3) $f^*|pH + qa(K_X + \Delta)| = |L| + \sum r_j F_j$

$\quad \overset{\shortparallel}{B(p,q)}$

bpf

fixed part

Step 8

$$\sum (-c r_j + a_j - p_j) F_j = A' - F$$

pick the c, p_j s.t.

F integral, reduced, part of fixed locus of $f^* B(p,q)$

$\lceil A' \rceil$ effective

f-exceptional

$$N(p', q') = f^* B(p', q') + A' - F - K_Y$$

this is ample when $p', q' \gg 0$ inside the strip

$$0 < aq' - rp' < aq - rp < 1$$

by the base point free method,

$$H^0(Y, f^*B(p',q') + \lceil A' \rceil)$$

$$(**) \longrightarrow\!\!\!\!\!\rightarrow H^0(F, f^*B(p',q')|_F + \lceil A' \rceil|_F)$$

$$\neq 0$$

<u>step 4</u>

$$N(p',q')|_F = (f^*B(p',q') + A' - F - K_Y)|_F$$

$$\underset{\text{ample}}{} \qquad \underline{f^*B(p',q')|_F} + A'|_F - K_F$$

so by step 2

$$P_F(x,y) = \chi(F, f^*B(p',q') + \lceil A' \rceil)_F)$$

$$\neq 0$$

+

$$h^i(F, f^*B(p',q')|_F + \lceil A' \rceil|_F) = 0$$

for $i > 0$

By step 3, there are infinitely many $p', q' \gg 0$

in $0 < aq' - rp' < aq - rp < 1$
$\phantom{0 < aq' - rp' <\ } \varepsilon$

$$P_F(p', q') \neq 0$$
$$\parallel$$
$$H^0(F, f^*B(p', q')|_F + \lceil A' \rceil|_F)$$

Step 10

by surjectivity of $(**)$

$$\Longrightarrow H^0(Y, f^*B(p', q') + \lceil A' \rceil) \neq 0$$
$$\parallel \qquad \ni s \qquad \text{s.t. } s|_F \neq 0$$
$$H^0(X, B(p', q'))$$

so

we have a section that doesn't vanish on $f(F) \not\subseteq L(p', q')$
$$\subsetneq L(p, q) = L_0$$

Contradiction so $r \in \mathbb{Q}$