Thm (Cone theorem)

read the section on bend+break

let (X,Δ) be a projective klt pair with Δ effective. Then:

1) there are countably many rational curves $C_i \subseteq X$ s.t. $0 < -(K_X+\Delta).C_i \le 2\dim X$ &

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

2) for any $\varepsilon > 0$ & H ample

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta+\varepsilon H\geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i]$$

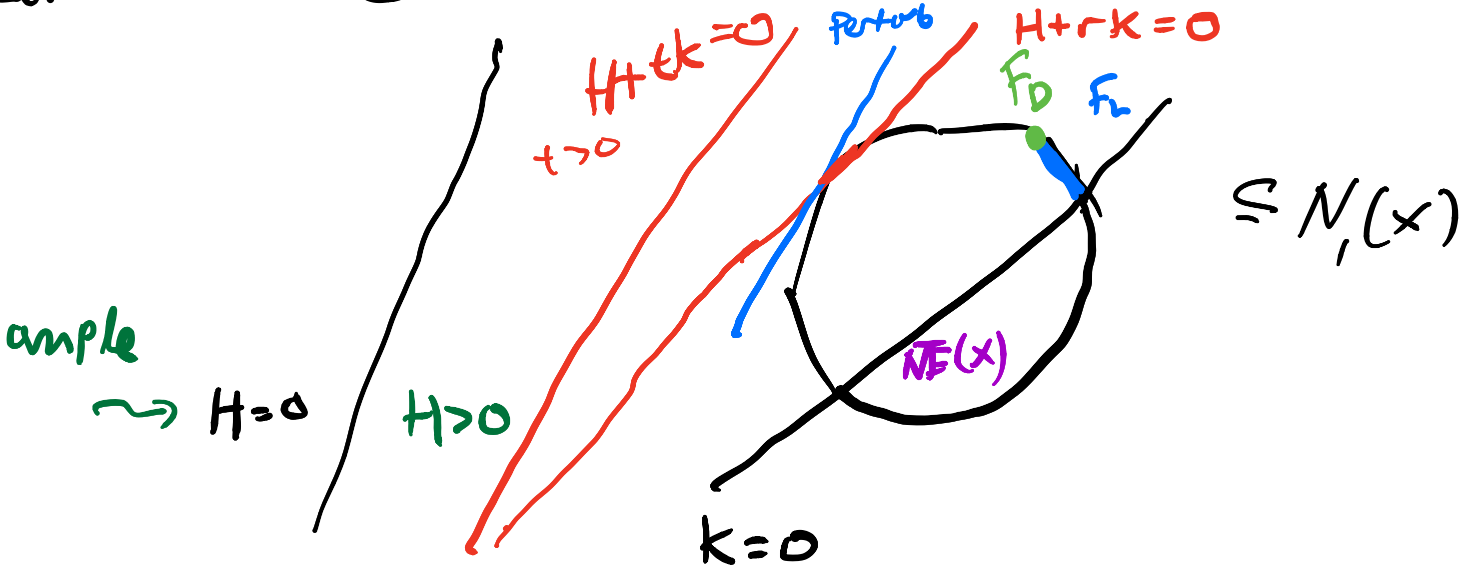
3) for $F \subseteq \overline{NE}(X)$ an $(K_X+\Delta)$-negative extremal face, $\exists!$ $\varphi_F : X \to Z$ which is projective s.t. i) $\varphi_{F*}\mathcal{O}_X = \mathcal{O}_Z$ ii) $\varphi_F(C) = pt \iff [C] \in F$

4) if L is a line bundle on X s.t. $L.C = 0$ for $[C] \in F$, then $L = \varphi_F^* L_Z$ for $L_Z \in Pic(Z)$

Proof $K := K_X + \Delta$, $a(K_X + \Delta)$

recall denominators in rat'l theorem are bounded by $a(\dim X + 1)$

let m be a common denominator



Suppose K is not nef.

<u>Step 1</u> L any nef but not ample divisor

$0 \neq F_L = L^{\perp} \cap \overline{NE}(X)$, suppose that

$$F_L \not\subseteq \overline{NE}_{K \geq 0}$$

$$r_L(n, H) = \max\{t \in \mathbb{R} \mid nL + H + \frac{t}{m}K \text{ nef}\}$$

$\mathbb{Z} \ni$ by rationality

$r_L(n,H) \leq r_L(n+1,H)$

Pick $\xi \in F_L \setminus^{\overline{NE}_{K\geq 0}} \underbrace{\left(nL + H + \frac{r_L(n,H)}{m} K\right)}_{\text{nef}} . \xi \geq 0$

uniform

$H.\xi > 0$

$K.\xi < 0$

$L.\xi = 0$

$r_L(n,H) \leq m \frac{H.\xi}{K.\xi}$

$\Longrightarrow$ the sequence $r_L(n,H)$ stabilizes to $r_L(H)$ for $n \geq n_0$

take $n > n_0$

$\underset{\text{nef}}{D(n,L,H)} = mnL + mH + r_L(H)K$

$\Downarrow$

$D(n-1,L,H)$ is also nef

$L|_{F_{D(n,L,H)}} = 0 \Longrightarrow F_{D(n,L,H)} \subseteq F_L \neq 0$

$+ \; F_{D(n,L,H)}^{0*} \setminus \{0\} \subseteq \overline{NE}(X)_{K<0}$

b/c $L|_{F_D} = 0$, $H|_{F_D \setminus 0} > 0 \Longrightarrow K|_{F_D \setminus 0} < 0$

<u>Step 2</u> Suppose $\dim F_L > 1$, then

<u>claim</u> we can pick H s.t.

$$\dim F_{D(n,L,H)} < \dim F_L$$

Pick some ample basis $\{H_1, \dots, H_p\}$ for $N^1(X)$

$$D(n,L,H_i)\Big|_{F_L} = \underbrace{(mH_i + r_L(H_i)K)}_{(\operatorname{Span} F_L)^*}\Big|_{F_L}$$

the H_i are linearly independent

so if $(\operatorname{Span} F_L)$ has $\dim > 1$

$\Rightarrow$ $D(n,L,H_i)\big|_{F_L}$ can't identically vanish for all i

$\Rightarrow \exists$ $D(n,L,H_i)\big|_{F_L} \not\equiv 0$

so

$$F_{D(n,L,H_i)} \subsetneq F_L$$

by repeatedly applying this process, we cut down to rays

F_L $\quad L$ nef $\quad \dim F_L = 1$

$F_L \setminus \{0\} \subseteq \overline{NE}(x)_{K<0}$

each face $F_{L'}$ for L' nef with $F_{L'} \not\subseteq \overline{NE}(x)_{K\geq 0}$ contains such rays

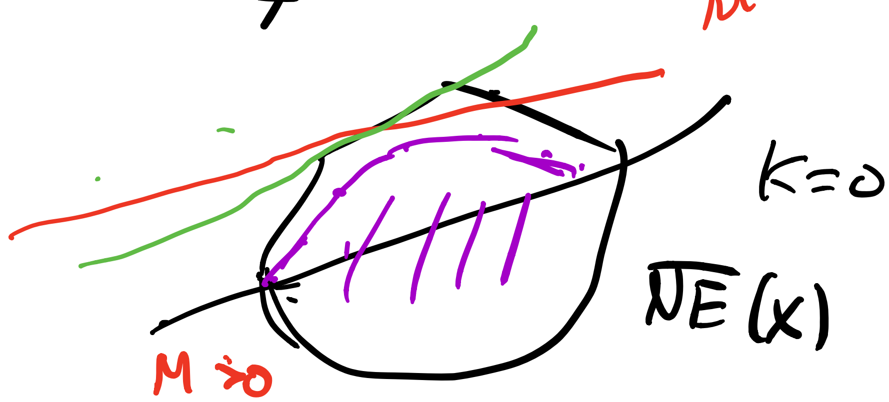
Step 3

$$\overline{NE}(x) = \underbrace{\Big(\overline{NE}(x)_{K\geq 0} + \sum_{\dim F_L} F_L\Big)}_{w}$$ closure

$\supseteq$ by definition

Suppose that $w \subsetneq \overline{NE}(x)$

then there exists rational M s.t.



$W \subseteq M_{>0}$, $\quad M_{<0} \cap \overline{NE}(X) \neq \emptyset$

apply rationality theorem to perturb M by some $H + tK$ so that we're on the boundary of $\overline{NE}(x)$ avoiding W

by the previous step, we can cut out a ray of $\overline{NE}(X)$ of the form F_L, but not contained in W, contradiction.

Step 4 there are no accumulation points of F_L inside $K < 0$

$F_L = \mathbb{R}_{\geq 0} \xi_L$, $\leftarrow$ integral

pick amples H_i s.t. $\{K, H_1, \ldots, H_{p-1}\}$ is a basis for $N^1(X)$

$$\frac{H_i \cdot \xi_L}{K \cdot \xi_L} = \frac{\text{integer}}{m} \qquad (*)$$

$\xi \mapsto (K \cdot \xi, H_1 \cdot \xi, \ldots, H_{p-1} \cdot \xi)$ gives coordinates for $N_1(X)$

if we look at $\mathbb{P}(\overline{N_1(X)})$

$\cup$

$\mathbb{A}^{p-1}_{\mathbb{R}} = \{K \neq 0\}$

$(**)$ give coordinates for the affine chart

$U = (K < 0)$

$$U \xrightarrow{g} \mathbb{A}^{p-1}$$

quotient by $\mathbb{R}_{>0}$

so $(*)$ tells us that F_L maps to a point with coordinates in $\frac{1}{m}\mathbb{Z}$ under g

$[F_L] \in \mathbb{P}(N_1(X))$

$\Longrightarrow$ Can't accumulate inside $\mathbb{A}^{p-1}$

$\Longrightarrow$ F_L can't accumulate inside of U

$$V = \mathbb{P}(\overline{NE}(X)_{K+\varepsilon H \leq 0}) \subseteq \mathbb{P}(U) = \mathbb{A}^{p-1}$$

for H ample

compact

so finitely many $[F_L]$ lie inside V

$\Rightarrow$ finitely many of the F_L lie inside $(K + \varepsilon H \leq 0)$

$$\overline{NE}(X)_{K\geq 0} + \sum_{\dim F_L = 1} F_L \subseteq \overline{NE}_{K+\varepsilon H \geq 0} + \sum_{F_L \cdot (K+\varepsilon H)<0} F_L \subseteq \overline{NE}(X)$$

$= \quad W \quad \subseteq$

finite sum $\Rightarrow$ closed

so $$\overline{NE}_{K+\varepsilon H \geq 0} + \sum_{F_L(K+\varepsilon H)<0} F_L = \overline{NE}(X)$$

Step 5 Use a limit argument ($\varepsilon \to 0$) to show that

$$\overline{NE}(X)_{K\geq 0} + \sum_{\dim F_L = 1} F_L$$

is closed

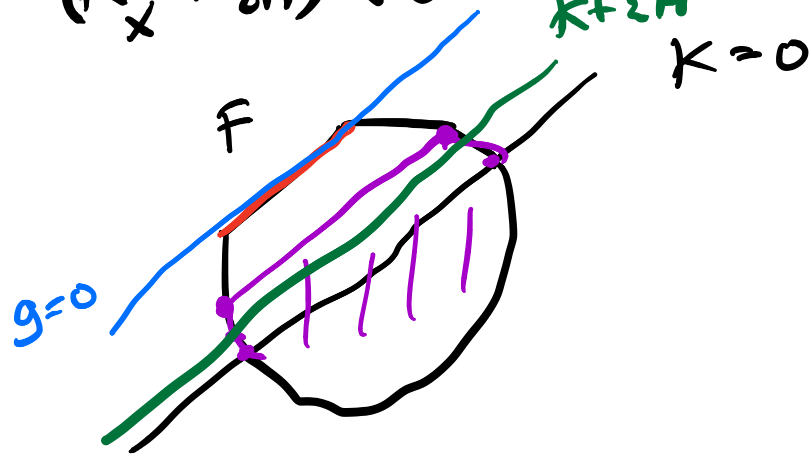
Step 6 Claim: if F any K-negative extremal face, then $F = F_D$ for D nef + Cartier

$V = \operatorname{Span}(F)^{\perp} \subseteq N^1(X)$

F is rational $\Rightarrow$ V is a rational subspace

Pick small enough $\varepsilon > 0$ s.t.

$$F \setminus \{0\} \subseteq (K_X + \varepsilon H) < 0$$



$$W = \overline{NE}_{K+\varepsilon H \geq 0} + \sum F_L$$

$$F_L \not\subseteq F$$

Pick $g \in V$ s.t. $\underline{g|_W > 0}$

open condition in V

since V is rational

$\Longrightarrow \exists$ g' with rat'l coordinates s.t. $g'|_F = 0$ $g'|_W > 0$

$g' = \left[\frac{1}{m} D\right]$ for D an integral divisor

so $D|_F = 0$, $D|_W > 0$

$\Longrightarrow$ D is nef, $D^\perp \cap \overline{NE}(X) = F$

Step 7 $\quad mD - (K_X + \Delta) + D$ nef

≥ 0 $\quad$ <0 on $F \setminus \{0\}$

$=0$ on F
>0 else

So for large m, $mD - (K_X + \Delta)$ is ample

So by base point free theorem,

for $b \gg 0$ $\quad \varphi_{|bD|} : X \to Z$ $\quad$ (Iitaka fibrations)

Stein factorize + take b large

call this Iitaka fibration

$\varphi_F : X \to Z$ $\quad$ projective with $\varphi_{F*} \mathcal{O}_X = \mathcal{O}_Z$

$\varphi_F(C) = 0 \iff C \cdot D = 0$ by semiample

$\iff [C] \in F$

So φ_F is as in part c) of the theorem, but this uniquely determines φ_F.