MMP for surfaces

X smooth projective surface

$$X = X_0 \to X_1 \to \cdots X_m$$

K_{X_m} is nef $\iff \kappa(X) \geq 0$

X_m is a minimal surface

K_{X_m} not nef

X_m is a Fano fibration $\iff \kappa(X) = -\infty$

X_i is smooth

$X_i = Bl_{p_i} X_{i+1}$

K_{X_m} - nef: K_{X_m} is semi-ample

$$\phi = \phi_{|dK_{X_m}|} : X_m \to Z \subseteq \mathbb{P}^N$$

$$Z = \operatorname{Proj} R(K_X) \qquad \dim Z = \kappa(X)$$

$\kappa = 2$: ϕ is birational, K_Z ample

Z the canonical model (X_m the minimal model)

Z has mild singularities

$\underline{\kappa = 1}$: $\phi : X_m \to Z \hookleftarrow$ smooth curve

with genus 1 fibers (elliptic fibrations)

$\underline{\kappa = 0}$: $\phi : X_m \to pt$ & X_m is K-trivial

i.e. $dK_{X_m} \sim 0$

$\underline{\kappa = -\infty}$ e.g. K_{X_m} is not nef

1) $X_m \to C$ a ruled surface

$$\begin{tikzcd} X_m \arrow[r] \arrow[d, phantom, "\wr"] & C \\ \mathbb{P}_C(\mathcal{E}) \arrow[ur] & \end{tikzcd}$$

(Fano fibration over a curve)

2) $X_m = \mathbb{P}^2$ (Fano variety i.e. Fano fibration over a pt)

§1: Curves on Surfaces (V.1 Hartshorne)

X smooth proj surface

C a curve in X $\quad$ D is a Cartier divisor

$$D.C = \deg \mathcal{O}_X(D)|_C \in \mathbb{Z}$$

Prop: This product extends to a unique pairing

$$\mathrm{Div}(X) \times \mathrm{Div}(X) \longrightarrow \mathbb{Z}$$
$$(D, C) \longmapsto D.C$$

1) if D & C intersect transversely
effective
then $D.C = \# D \cap C$

2) The product depends only on $\sim$ classes $\Rightarrow$ $\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}$

Adjunction: $D \subseteq X$ is a curve then

$$(K_X + D)|_D = K_D$$

$$\omega_X(D)|_D = \omega_D$$

Riemann-Roch:

constant depending on X

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2} D.(D-K_X) + \chi(\mathcal{O}_X)$$

$$\| $$

$$h^0(X,\mathcal{O}_X(D)) - h^1(X,\mathcal{O}_X(D)) + h^2(X,\mathcal{O}_X(D))$$

Num equivalence:

$D \equiv D' \iff D.C = D'.C$ for any curves C

$$\mathrm{Pic}(X)/_{\equiv} \otimes \mathbb{R} \; (\otimes \mathbb{Q}) = N^1(X) \; {}_{\mathbb{Q}}$$

$N^1(X) \otimes N^1(X) \to \mathbb{R}$ nondegenerate

$\rho(X) := \dim_{\mathbb{R}} N^1(X) < \infty$ Picard rank

Hodge Index Theorem

The intersection pairing on $N^1(X)$ has signature $(1, \rho-1)$

+ $\quad$ −

Cone of Curves:

$$NE(X) = \left\{ \sum a_i [C_i] \;\middle|\; \begin{array}{l} a_i \geq 0 \\ C_i \text{ is an irred curve} \end{array} \right\}$$

$[C]$ = numerical eq class

$\overline{NE}(X)$ = closure of $NE(X) \subseteq N^1(X)$

A divisor L is <u>nef</u> if $L.C \geq 0$ for any curve C (effective)

<u>Nef cone</u> is the dual cone under intersection to $\overline{NE}(X)$

<u>Thm</u> (Kleiman's criterion)

L ample $\iff$ $L \cdot > 0$ on $\overline{NE}(X) \setminus \{0\}$

<u>Thm</u> (Nakai-Moishezon)

L is ample $\iff$ $L.C > 0$ for C effective curves

$L^2 > 0$

Fact

Suppose L is semi-ample

dL is base point free

$\phi = \phi_{|dL|} : X \to Z \subseteq \mathbb{P}^N$ $\quad dL = \phi^* H$

$C \subseteq \phi^{-1}(p) \iff L.C = 0$

1) If ϕ is birational, $E \subseteq \phi^{-1}(p)$

$\Rightarrow$ $E^2 < 0$ (exceptional curves)

2) $\phi : X \to Z \leftarrow$ curve

$D \subseteq \phi^{-1}(P) \Rightarrow D^2 \leq 0$

$D^2 = 0 \iff D = mF$

$F = \phi^{-1}(P)$

§2: Blowups of smooth surfaces

$$X' := Bl_p X \xrightarrow{\mu} X \ni p$$

smooth

$$\overset{\shortparallel}{\operatorname{Proj} \bigoplus_{d \geq 0} m_p^d}$$

Facts (V.3 H)

$E = \mu^{-1}(p)$

1) $E \cong \mathbb{P}^1$

2) $E^2 = -1$

$N_{E/X} = \mathcal{O}_{\mathbb{P}}(-1)$

3) $\mu : X' \setminus E \xrightarrow{\cong} X \setminus p$

4) $\mathrm{Pic}(X') \underset{\mu^*}{\cong} \mathrm{Pic}(X) \oplus \mathbb{Z}E$

$\rho(X') = \rho(X) + 1$

5) $K_{X'} = \mu^* K_X + E$ (adjunction + intersection)

Def $E \subseteq X$ is a (-1)-curve if $E \cong \mathbb{P}^1$ + $E^2 = -1$

Exc: E is a (-1)-curve $\iff$ $E^2 < 0$, $K_X . E < 0$

Thm (Castelnuovo's Contraction Theorem)

X smooth proj surface
$\cup\!\!\shortmid$
E is a (-1)-curve

<u>Then:</u> $\exists\ \mu : X \to X_1$ proper birational

s.t. $\mu(E) = P$, X_1 is smooth

μ is the blowup of X_1 at P

<u>Proof</u> Want an L s.t.

$\varphi_{|L|}$ is a morphism & $\varphi_{|L|} = \mu$

$L.E = 0$ & $L.C$ is not zero for any other curves

A very ample

$A.E = k > 0$ $E^2 = -1$

$\Rightarrow$ $L = A + kE$ satisfies $L.E = 0$

also $L.C > 0$ b/c $A.C > 0$
$E.C \geq 0$

$$L_n = A + nE \qquad n = 0, \dots, k$$

Step 1

$$H^1(X, \mathcal{O}_X(L_n)) = 0$$

for all $0 \le n \le k$

$n = 0$ $\quad L_n = A$

wlog $\quad H^1(X, \mathcal{O}_X(A)) = 0$

by some vanishing

$n > 0$ $\quad$ induct

$$0 \to \mathcal{O}_X(A + \underline{(n-1)E}) \to \mathcal{O}_X(A+nE) \to \mathcal{O}_E(A+nE) \to 0$$

$$(0 \to \mathcal{O}_X(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0) \otimes \mathcal{O}_X(L_n)$$

$$0 \to \cancel{H^1(\mathcal{O}_X(L_{n-1}))} \to \cancel{H^1(\mathcal{O}_X(L_n))} \to \cancel{H^1(\mathcal{O}_E(A+nE))}$$

$$0 \qquad\qquad\qquad 0 \qquad\qquad\qquad 0$$

$\Rightarrow$

$$E = \mathbb{P}^1 \qquad \mathcal{O}_E(A+nE) = \mathcal{O}_{\mathbb{P}^1}(\underset{\ge 0}{k-n})$$

Step 2

$|A + nE|$ is bpf for all $0 \le n \le k$

$Bs(A+nE) \subseteq E$

$\underline{|A| + nE} \subseteq |A+nE|$

Want to show no base points in E

$$H^0(\mathcal{O}_X(A+nE)) \twoheadrightarrow H^0(\mathcal{O}_E(A+nE)) \to H^1$$

$H^1 = 0$

$\forall p \in E$ $\exists s$ s.t. $s(p) \neq 0$

$\mathcal{O}_E(A+nE) = \mathcal{O}_{\mathbb{P}^1}(k-n)$, $k-n \geq 0$

$\Longrightarrow$ $A+nE$ is bpf

Step 3: $\phi = \phi_{|A+kE|} : X \to X_1 \subseteq \mathbb{P}^N$

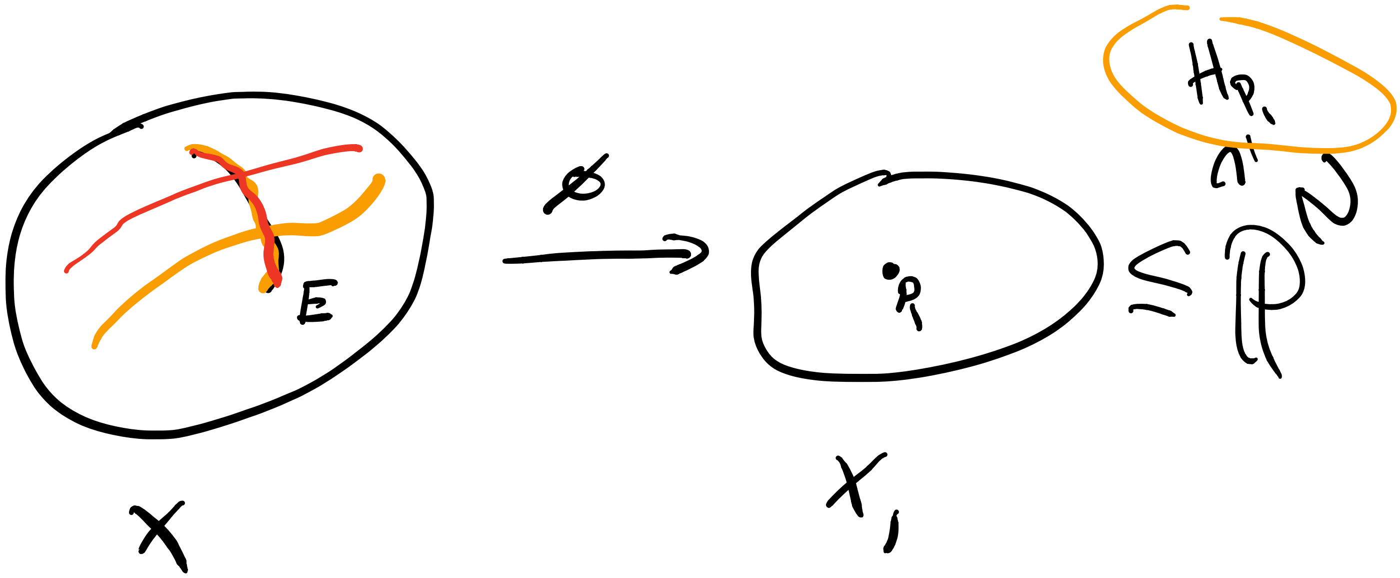
ϕ is an isomorphism away from E

b/c it contains $|A|+nE$ w/ is very ample away from E

+ $\phi(E) = p_1$

+ $\phi^{-1}(p_1) = E$

$|A+(k-1)E| + E \subseteq |A+kE|$



Step 4

$$\nu : X_0 \to X_1$$

$$\begin{tikzcd} & X_0 \arrow[d] \\ X \arrow[ur, "f"] \arrow[r, "\phi"'] & X_1 \end{tikzcd}$$

f is an isomorphism away from E

$f(E) = p$

$\underline{f^{-1}(p) = E}$

f is proper birational between normal varieties

$$\Longrightarrow \quad f_* \mathcal{O}_X = \mathcal{O}_{X_0}$$

Step 5 X_0 is smooth at p

Thm on Formal Functions (III.11.1 H)

$$\hat{\mathcal{O}}_{X_0, p} = (f_* \mathcal{O}_X)^\wedge_p = \varprojlim H^0(X, \mathcal{O}_{E_n})$$

$$E_n = f^{-1}\left(\operatorname{Spec} \mathcal{O}_{X,P}/\mathfrak{m}_P^n\right)$$
$$= V(\mathfrak{m}_P^n \mathcal{O}_X)$$

$V(I_E^n)$

$$\varprojlim H^0(X, \mathcal{O}_{V(\mathfrak{m}_P^n \mathcal{O}_X)}) = \varprojlim H^0(X, \mathcal{O}_{V(I_E^n)})$$

$$\overset{\shortparallel}{E_n}$$

<u>$n=1$</u> $\quad H^0(X, \mathcal{O}_E) = H^0(E, \mathcal{O}_E) = k$

E is (-1)-curve $\qquad I_E = I$

$$N_{E/X} = \mathcal{O}_{\mathbb{P}^1}(-1) \Rightarrow I/I^2 = \mathcal{O}_{\mathbb{P}^1}(1)$$

$+$ E is Cartier $\Rightarrow$ I locally generated by one element

$$\begin{array}{ccc} I^n/I^{n+1} \ni f^n & \cong & \mathcal{O}_{\mathbb{P}^1}(n) \\ \uparrow\cong \qquad \uparrow & & \\ (I/I^2)^n \ni f & \cong & \mathcal{O}_{\mathbb{P}^1}(1) \end{array}$$

$\Rightarrow H^1(I^n/I^{n+1})$

$$0 \to I^n/I^{n+1} \to \mathcal{O}_{E_{n+1}} \to \mathcal{O}_{E_n} \to 0$$

$$0 \to H^0(I^n/I^{n+1}) \to H^0(\mathcal{O}_{E_{n+1}}) \to H^0(\mathcal{O}_{E_n}) \to 0$$

<u>n=1</u>

$$0 \to \langle x, y\rangle \longrightarrow \frac{k[x,y]}{(x,y)^2} \longrightarrow k \to 0$$

by induction

$$0 \to \underset{\substack{\| \\ \langle \text{forms of degree } n\rangle}}{H^0(\mathcal{O}_{\mathbb{P}^1}(n))} \to \frac{k[x,y]}{(x,y)^{n+1}} \to \frac{k[x,y]}{(x,y)^n} \to 0$$

$$\Longrightarrow \hat{\mathcal{O}}_{X_0, p} = \varprojlim_n \frac{k[x,y]}{(x,y)^n} = k[\![x,y]\!]$$

$\Longrightarrow$ X_0 is smooth at p

<u>Step 6</u>

$$\begin{tikzcd} & Bl_p X_0 \arrow[d] \\ X \arrow[ur, "\mu"] \arrow[r, "f"] & X_0 \end{tikzcd}$$

Check that μ is an iso

$$\mu_* \mathcal{O}_X = \mathcal{O}_{Bl_p X_0}$$

<u>Def</u> X is a minimal surface if it contains no (-1) curves

<u>Cor</u> For any smooth projective surface $\exists$ a finite sequence of blowups

$$X = X_0 \to X_1 \to \dots \to X_m$$

s.t. 1) $X_i = Bl_p X_{i+1}$

2) X_i is smooth

3) X_m is minimal

<u>Pf</u> if X_i has a (-1)-curve

Castelnuovo $\Rightarrow$ $X_i \to X_{i+1}$

$$\rho(X_i) = \rho(X_{i+1}) + 1 > 0$$

classify X_m based on whether

K_{X_m} is nef or not