Thm (Cone theorem)

read the section on bend+break

Let (X,Δ) be a projective klt pair with Δ effective. Then:

1) there are countably many rational curves $C_i \subseteq X$ s.t. $0 < -(K_X+\Delta).C_i \leq 2\dim X$ &

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i]$$

2) for any $\varepsilon > 0$ & H ample

$$\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta+\varepsilon H \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i]$$

3) for $F \subseteq \overline{NE}(X)$ an $(K_X+\Delta)$-negative extremal face, $\exists!$ $\varphi_F : X \to Z$ which is projective s.t. i) $\varphi_{F*}\mathcal{O}_X = \mathcal{O}_Z$ ii) $\varphi_F(C) = pt \iff [C] \in F$

4) if L is a line bundle on X s.t. $L.C = 0$ for $[C] \in F$, then $L = \varphi_F^* L_Z$ for $L_Z \in \mathrm{Pic}(Z)$

Proof (Continued)

$K = K_X + \Delta$

recap
$$\overline{NE}(X) = \overline{NE}(X)_{K \geq 0} + \sum_{\dim F_L} F_L$$

rays

L = nef, not ample Cartier

$$F_L = L^\perp \cap \overline{NE}(X)$$

s.t. $F_L \setminus \{0\} \subseteq \overline{NE}(X)_{K<0}$

& for $\varepsilon > 0$, H ample

$$\overline{NE}(X)_{K+\varepsilon H \geq 0} + \sum_{F_L \cdot (K+\varepsilon H) < 0} F_L$$

finite

+ for any extremal face F s.t. $F \setminus \{0\} \subseteq \overline{NE}(X)_{K \leq 0}$, then there is a nef divisor D s.t. $F = F_D$

Then, by base point free theorem D is semiample

$aD - (K_X + \Delta)$ is ample

$\Rightarrow \varphi_{|bD|}$ morphism for large b

Iitaka fibration for $b \to \infty$

+ Stein factorization

$\varphi_F : X \to Z$ projective morphism

s.t. Z is normal

i) $\varphi_{F_*} \mathcal{O}_X = \mathcal{O}_Z$

ii) $\varphi_F(C) = pt \iff [C] \in F$

uniquely characterize φ_F

<u>Step 8</u>

let $F = R = F_L$ is a <u>ray</u>

If φ_R is not the identity

since $R \neq \{0\}$

then φ_R contracts some curve C

$\Rightarrow [C] \in R$

so $R = F_L = \mathbb{R}_{\geq 0}[C]$

<u>Step 9</u> Suppose D is nef s.t.

$F_D = F$

uniqueness of $\varphi_F \Rightarrow$ the Iitaka fibration of the bpf series $|bD|$

so $D = \varphi_F^* D_Z$ for some D_Z

Now suppose L is any divisor s.t. $L.C = 0$ for $[C] \in F$

Consider $mD + L$ $m \gg 0$ with D as above

$(mD+L).C = 0$ for $[C] \in F$

$D.\xi > 0$ for $\xi \in \overline{NE}(X) \setminus F$

so $mD + L$ is nef for $m \gg 0$

so by above argument,

$$mD + L = \varphi_F^* D_Z'$$

$$L = \varphi_F^* (D_Z' - m D_Z) \qquad \boxed{/\!/}$$

<u>Cor</u> if $F = R = \mathbb{R}_{\geq 0}[C]$ is a $(K_X + \Delta)$-negative extremal ray, then the following sequence is exact

$\rho(Z) = \rho(X) - 1$

$$0 \to \operatorname{Pic}(Z) \xrightarrow{\varphi_F^*} \operatorname{Pic}(X) \to \mathbb{Z}$$
$$L \mapsto L.C$$

<u>Proof</u> just need to check that φ_F^* is injective, since exactness in the middle is step 9.

$$\varphi_{F*}\varphi_F^*\mathcal{O}_Z(L) = \mathcal{O}_Z(L)\otimes\varphi_{F*}\mathcal{O}_X$$
$$= \mathcal{O}_Z(L)$$
$$\Longrightarrow \varphi_F^* \text{ is injective}$$

<u>Cor</u> (Classification of extremal contractions)

Suppose (X,Δ) is a projective $\mathbb{Q}$-factorial klt pair & R is a $(K_X+\Delta)$-negative extremal ray. Then $\varphi_R : X \to Z$ is either

1) birational with exceptional locus $E \subseteq X$ an irreducible divisor

2) small birational map

3) $\dim Z < \dim X$ & in this case φ_R is a log Fano fibration

Say F is a generic fiber of φ_R in case 3), $(K_X+\Delta)|_F = K_F + \Delta|_F$

$\Rightarrow (X, \Delta_F)$ is a pair with $-(K_F+\Delta|_F)$ ample

Cor let (X,Δ) be a projective klt pair, $R \subseteq \overline{NE}(X)$ a $(K_X+\Delta)$-negative extremal ray. Suppose X is $\mathbb{Q}$ fact & φ_R is either divisorial or a fibration (Mori fiber space)

then Z is $\mathbb{Q}$-factorial

Proof 1) φ_R is divisorial, $E \subseteq X$ exceptional divisor

$E.R < 0$. let B be a weil divisor on Z. $R = \mathbb{R}_{\geq 0}[C]$

$$(\varphi_{F*}^{-1} B + sE).C = 0$$

there exists an s s.t.

since X is $\mathbb{Q}$ factorial, then

$m(\varphi_{F*}^{-1} B + sE)$ is Cartier

$\Downarrow$

$\varphi_F^* M_Z$ $\quad$ $M_Z \in Z$

$B \sim_{\mathbb{Q}} \frac{1}{m} M_Z \Rightarrow B$ is $\mathbb{Q}$-cartier

since exceptional locus of φ_F is codim 2 on Z

3) if $\dim Z < \dim X$, B weil div on Z

$B^\circ \subseteq B$ cartier locus $\quad$ big open $\cup$ U

$\| $

$U \cap B$ $\quad$ $\overline{\varphi_F|_{\varphi_F^{-1}(U)}^* B^\circ} = D \subseteq X$

mD is cartier for some m

but mD. generic fiber $= 0$

so $mD.C = 0$ for $[C]$ spanning the ray

$\Rightarrow mD = \varphi_F^* M_Z$ $\quad$ $M_Z \in \mathrm{Pic}(Z)$

$B \sim_{\mathbb{Q}} M_Z/m \Rightarrow B$ is $\mathbb{Q}$-cartier. $\square$

Cor $\varphi_F : X \to Z$ extremal contraction of a $(K_X+\Delta)$-negative face

(X,Δ) proj klt,
$\Delta \geq 0$

Then $R^i \varphi_{F*} \mathcal{O}_X = 0$ for $i > 0$

$$\boxed{R\varphi_{F*} \mathcal{O}_X \simeq_{q\text{is}} \mathcal{O}_Z}$$

Proof $-(K_X+\Delta)$ is φ_F-ample

$D = 0$ so by KV vanishing

$$R^i \varphi_{F*} \mathcal{O}_X(D) = R^i \varphi_{F*} \mathcal{O}_X = 0 \quad \text{for } i > 0$$

The relative setting

Thm (relative bpf theorem) let (X,Δ) a klt pair with $\Delta \geq 0$, $f : X \to Y$ projective morphism. D f-nef Cartier divisor such that

$$aD - (K_X + \Delta) \text{ is } f\text{-big \& } f\text{-nef}$$

for some $a > 0$

then, D is f-semiample

bD is f-bpf for $b \gg 0$

i.e. $f^* f_* \mathcal{O}_X(bD) \twoheadrightarrow \mathcal{O}_X(bD)$

<u>Proof</u> (sketch)

<u>Step 1</u> Compactify & resolve to reduce to the case of f a morphism of projective varieties

<u>Step 2</u> let A be an ample on Y

it suffices to show that

$$mD' = mD + mf^*A \text{ is}$$

base point free

indeed, for any $x \in X$, $s(x) \neq 0$

$$s \in H^0(X, mD') \xrightarrow{\sim} H^0(Y, f_*(mD + mf^*A))$$

$$\downarrow$$

$$H^0(U, f_* mD|_U) = H^0(V, mD|_V)$$

$$\begin{array}{ccccc} & & x & & \\ f^{-1}(U) & = & V & \subseteq & X \\ & & \downarrow & & \downarrow \\ f(x) & \in & U & \subseteq & Y \end{array}$$

Pick $U = \operatorname{Spec} R$ to be a trivialization of $\mathcal{O}_Y(A)$

<u>Step 3</u> $aD - (K_X + \Delta)$ is f-big

$aD - (K_X + \Delta) + f^*H$ big if

H is very ample enough

Pick some E effective s.t.

$aD - (K_X + \Delta) + f^*H - \varepsilon E$

is ample

For $\varepsilon > 0$ small, $(X, \Delta + \varepsilon E)$ is klt

$A = H/a$ $\quad D' = D + f^*A$

$(X, \underbrace{\Delta + \varepsilon E}_{\Delta'})$ is klt &

$aD' - (K_X + \Delta') = G$ is ample

So if D' is nef $\Rightarrow$ semiample by bpf

<u>Step 5</u> need to check that D'

can be made to be nef

apply cone theorem to (X, Δ')

$\Rightarrow \exists$ finitely many extremal rays R $= \mathbb{R}_{\geq 0}[C]$

s.t. $R \setminus \{0\} \subseteq (K_X + \Delta' + G)_{<0}$

$$\underbrace{K_X + \Delta' + G}_{= D'} \quad D' < 0$$

since $D' = f\text{-nef} + \text{pullback}$,

then $C \not\subseteq$ fibers of f

$+ \quad C \cdot f^* A = f_* C \cdot A > 0$

so by making A more positive, we can make $D' \cdot C \geq 0$

so by modifying A, we can make D' nef $\blacksquare$