Thm (relative cone theorem)

(X, Δ) is klt, $f: X \to Y$ a projective morphism of conn quasiprojective normal varieties. Then

1) $\overline{NE}(X/Y) = \overline{NE}(X/Y)_{K_X+\Delta \geq 0} + \sum_{\text{countable}} \mathbb{R}_{\geq 0}[C_i]$

2) for $\varepsilon > 0$, H f-ample,
$$\overline{NE}(X/Y) = \overline{NE}(X/Y)_{K_X+\Delta+\varepsilon H} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i]$$

3) contractions of $(K_X+\Delta)$-negative extremal faces of $\overline{NE}(X/Y)$ exist & are unique

4) line bundles that are $\equiv$ trivial on an extremal face F are pulled back by ~~the~~ extremal contraction

Proof sketch

Step 1 Compactify X, Y, f to be a projective morphism between projective varieties, and extend the divisors appropriately

step 2 $\xi \in \overline{NE}(X/Y)_{K_X+\Delta < 0}$

$\xi = \eta + \sum r_j [C_j]$ by cone theorem on (X, Δ)

$\overline{NE}(X)_{K_X + \Delta \geq 0}$ generators of $(K_X + \Delta)$-negative rays of $\overline{NE}(X)$

$$0 = f_* \xi = f_* \eta + \sum r_j \underline{f_* [C_j]} \in \overline{NE}(Y)$$

$r_j > 0$ $\quad f_* \eta \in \overline{NE}(Y)$

Since Y is projective, $\overline{NE}(Y)$ contains no lines $\Rightarrow$ $f_* \eta = f_* [C_j] = 0$

so $$\overline{NE}(X/Y) = \overline{NE}(X/Y)_{K_X + \Delta \geq 0} + \sum_{f_*[C_j] = 0} \mathbb{R}_{\geq 0} [C_j]$$

completes first part

<u>Step 3</u> let H be an f-ample

$\varepsilon > 0$, can pick some A ample on Y s.t. $\varepsilon H + f^* A$ is ample on X, so

$$\overline{NE}(X) = \overline{NE}(X)_{K_X + \Delta + \varepsilon H + \underline{f^* A \geq 0}} + \underline{\sum_{\text{finite}} \mathbb{R}_{\geq 0} [C_j]}$$

$\cup\!\!|$

$$\overline{NE}(X/Y)_{\underline{K_X + \Delta + \varepsilon H \geq 0}}$$

Step 4 use relative bpf theorem to produce extremal contractions

$$\begin{tikzcd} X \arrow[r, "\phi_R"] \arrow[dr, "f"'] & X' \arrow[d, "f'"] \\ & Y \end{tikzcd}$$

continue as before

Cor relative cone theorem holds for dlt pairs.

Proof if (X, Δ) is dlt, then

$(X, (1-\delta)\Delta)$ is klt for

$0 < \delta \ll 1$

H ample, $\varepsilon > 0$, pick δ s.t. $\varepsilon H + \delta\Delta$ is ample

$$\overline{NE}(X/Y) = \overline{NE}(X/Y)_{K_X + (1-\delta)\Delta + \varepsilon H + \delta\Delta} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i]$$

$$= \overline{NE}(X/Y)_{K_X + \Delta + \varepsilon H} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i]$$

+ argument as in the proof of the cone theorem to handle $\varepsilon \to 0$

The log minimal model program

(LMMP/ MMP)

(X, Δ)

recall when $\Delta = 0$, a <u>minimal model</u>

is an $\phi^m : X \dashrightarrow X^m$ s.t. X^m

i) X^m has terminal sing

ii) K_{X^m} nef

iii) ϕ^m is birational

$\phi^c : X \dashrightarrow X^c$ a <u>canonical model</u>

i) X^c has canonical sing

ii) K_{X^c} is ample

iii) ϕ^c is birational

$X^c = \mathrm{Proj}\ R(K_X)$ Abundance

LMMP

<u>Input</u> = $(X, \Delta) \xrightarrow{\pi} B$ projective morphism

i) (X, Δ) is dlt

ii) X is $\mathbb{Q}$-Factorial

<u>Algorithm</u> Pick a $(K_{X_i} + \Delta_i)$-negative extremal

set $(X_0, \Delta_0) = (X, \Delta)$ ray $R \subseteq \overline{NE}(X/Y)$

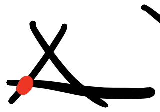
if ϕ_R is not a fiber space: let $\phi_i : X_i \dashrightarrow X_{i+1}$ be ϕ_R or its flip, $\Delta_{i+1} = \phi_{i*}\Delta_i$

Need to know X_{i+1} is $\mathbb{Q}$-factorial, (X_{i+1}, Δ_{i+1}) is dlt

Output either a Mori fiber space $\phi : X_n \to Z$ with $\dim Z < \dim X_n$

or (X_n, Δ_n) where $K_{X_n} + \Delta_n$ is nef

Rmk ϕ_i never extract divisors, i.e. ϕ_i^{-1} does not contract divisors.

Ex $(\mathbb{P}^2, \quad)$  $\xleftarrow{g} (Bl_p \mathbb{P}^2, D' = g_*^{-1} D)$

$\quad\quad\quad\quad$ = D

In this example, the pluricanonical sections are different

(X, D) s.t. $K_X + D$ is $\mathbb{Q}$-Cartier

Def A $(K_X + D)$-flipping contraction is a small birational morphism

$f: X \to Y$ s.t. $\rho(X/Y) = 1$ s.t.

$-(K_X + D)$ is f-ample.

The flip of f, $f^+ : X^+ \to Y$ s.t.

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

i) $\phi_* D =: D^+$ satisfy that $K_{X^+} + D^+$ is $\mathbb{Q}$-cartier

ii) f^+ is small

iii) $K_{X^+} + D^+$ is ample

Prop let (X, Δ) be a $\mathbb{Q}$-factorial dlt pair, $\phi_R : X \to Y$ a $(K_X + \Delta)$-flipping contraction of an extremal ray, let $\phi_R^+ : X^+ \to Y$ be a flip.

Then 1) X^+ is $\mathbb{Q}$-factorial

2) $\rho(X) = \rho(X^+)$

Proof

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "\phi_R"'] & & X^+ \arrow[dl, "\phi_R^+"] \\ & Y & \end{tikzcd}$$

φ is small birational, extracts no div

$$WDiv_{\mathbb{Q}}(X) \underset{\varphi_*^{-1}}{\overset{\varphi_*}{\cong}} WDiv_{\mathbb{Q}}(X^+)$$

D^+ some weil divisor on X^+

$\varphi_*^{-1} D^+ =: D$

$$(D + r(K_X + \Delta)).R = 0$$

there exists an r s.t.

$$\phi_R^* m D_Z \sim m(D + r(K_X + \Delta))$$

by cone theorem, where

mD_Z is Cartier

$$D^+ = \varphi_* D \sim \varphi_* \phi_R^* D_Z - \varphi_* r(K_X + \Delta)$$

$$(\phi_R^+)^* D_Z - r(K_{X^+} + \Delta^+)$$

$\Rightarrow$ D^+ is $\mathbb{Q}$-Cart $\mathbb{Q}$-Cartier

$\Rightarrow$

$$\mathrm{Div}_{\mathbb{Q}}(X) = \mathrm{WDiv}_{\mathbb{Q}}(X) = \mathrm{WDiv}_{\mathbb{Q}}(X')$$

$$= \mathrm{Div}_{\mathbb{Q}}(X')$$

by previous step

so $\rho(X) = \rho(X')$.

∎

Prop let (X, Δ) with Δ effective let $g: X \dashrightarrow X'$ be either a divisorial contraction of an extremal ray, or a flip of a small contraction of an extremal ray. Then

(X, Δ) is klt (resp lc, resp dlt)

$$\Downarrow$$

$(X', g_* \Delta)$ is klt (resp lc, resp dlt)

Proof **Lemma**

0) g, g' proj

1) $g_* \Delta = g'_* \Delta'$

2) $-(K_X + \Delta)$ is $\mathbb{Q}$-Cartier, g-nef

3) $K_X + \Delta$ is $\mathbb{Q}$-Cartier g'-nef

$$\begin{tikzcd} (X, \Delta) \arrow[rr, dashed, "\phi"] \arrow[dr, "g"'] & & (X', \Delta') \arrow[dl, "g'"] \\ & Y & \end{tikzcd}$$

Then for any exceptional E lying over Y, $a(E, X, \Delta) \leq a(E, X', \Delta')$

<u>Furthermore,</u> if 1) $-(K_X + \Delta)$ is g-ample and g is not an iso at the generic point of $\text{center}_Y(E)$

<u>OR</u>

2) $K_{X'} + \Delta'$ is g'-ample & g' is not an iso at the generic point of $\text{center}_Y(E)$

then $a(E, X, \Delta) < a(E, X', \Delta')$

<u>Pf of lemma</u> pick

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ X \arrow[rr, dashed, "\phi"] \arrow[dr, "g"'] & & X' \arrow[dl, "g'"] \\ & Y & \end{tikzcd}$$

$h = g \circ f = g' \circ f'$

suppose that

$\text{center}_Z(E) \subseteq Z$ is a divisor

$$K_Z - \sum a_i(E_i, X, \Delta) E_i = f^*(K_X + \Delta)$$

$$K_Z - \sum a_i(E_i, X', \Delta') E_i = f'^*(K_{X'} + \Delta')$$

$-B$

$$\sum a_i(E_i, X, \Delta) - a_i(E_i, X', \Delta') E_i$$

$$= f'^*(K_{X'} + \Delta') - f^*(K_X + \Delta)$$

so by our assumptions on the positivity of $K_{X'} + \Delta'$ & $-(K_X + \Delta)$, then $-B$ is h-nef

<u>Negativity lemma</u> $-B$ h-nef

$\Rightarrow$ B is effective $\Leftrightarrow$ $h_* B$ is effective

by negativity lemma,

$$a(E_i, X, \Delta) \leq a(E_i, X', \Delta')$$

under our extra assumptions, $-B|_E$ is not numerically trivial at the generic point

$\Rightarrow$ strict inequality

$\boxtimes$

This proves the prop for klt & lc

(X, Δ) dlt, pick $Z \subseteq X$ s.t. $(X \setminus Z, \Delta \setminus Z)$ is snc

let $Z' = \phi(Z) \cup Ex(g^{-1})$
$\subseteq X'$

$X' \setminus Z' \subseteq X \setminus Z$ on $X' \setminus Z'$ we have an snc divisor

let E be a divisor with $center_{X'}(E) \subseteq Z'$

lemma

dlt

$a(E, X', \Delta') \geq a(E, X, \Delta) \geq -1$

if $center_{X'}(E) \subseteq Ex(g^{-1})$ then the 1^{st} inequality is strict

if $center_{X'}(E) \subseteq \phi(Z) \Rightarrow center_X(E) \subseteq Z$

so the 2^{nd} inequality is strict $\blacksquare$