Types of models

$f: (X, \Delta) \to B$
lc pair proj over B

Def

(X, Δ) is a log minimal model $/B$
if (X, Δ) has dlt singularities &
$K_X + \Delta$ is f-nef

What should be called the log min/can model of the pair (X, Δ)?

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

1) g projective

2) ϕ extracts no divisors

$\Delta_Y = \phi_* \Delta$

e.g. if ϕ a morphism

$K_X + \Delta = \phi^*(K_Y + \Delta_Y) + \underbrace{\sum a_i E_i}_{\text{effective exceptional}}$

3) $a(E, X, \Delta) \leq a(E, Y, \Delta_Y)$ for all ϕ-exceptional divisors E

3) $\Rightarrow$ $a_i \geq 0$

3') $\Rightarrow$ $a_i > 0$

3') $a(E, X, \Delta) < a(E, Y, \Delta_Y)$
for E ϕ-exc

Def $f: (X, \Delta) \to B$ lc pair proj$/B$

a pair $(Y, \Delta_Y = \phi_* \Delta)$

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

is a

I) weak log canonical model $WLCM(X, \Delta/B)$ if it satisfies 1+ 2+ 3 above & $K_Y + \Delta_Y$ is g-nef

II) (log) minimal model (log terminal model $LTM(X, \Delta/B)$) if (X, Δ) is dlt, it satisfies 1+ 2 + 3' $K_Y + \Delta_Y$ is g-nef

III) log canonical model if its a $WLCM(X, \Delta/B)$ & $K_Y + \Delta_Y$ is g-ample

<u>Prop</u> $f: (X, \Delta) \to B$ as above, $(Y, \Delta_Y) = WLCM(X, \Delta/B)$ then for <u>all</u> E lying over X, $a(E, X, \Delta) \leq a(E, Y, \Delta_Y)$

<u>Proof</u>

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

s.t. $\mathrm{Center}_W(E)$ is a divisor

$$a(E,Y,\Delta_Y) - a(E,X,\Delta)$$
$$\parallel$$
$$\mathrm{coeff}_E(A_Y - A_X)$$

$$K_W = p^*(K_X+\Delta) + A_X$$

$$K_W = q^*(K_Y+\Delta_Y) + A_Y$$

$$B := p^*(K_X+\Delta) - q^*(K_Y+\Delta_Y) = A_Y - A_X$$

1) $P_* B$ is effective by assumption

2) $-B = q^*(K_Y+\Delta_Y) - p^*(K_X+\Delta)$

is p-nef

Indeed if $\xi \in \overline{NE}(W/X)$ $\quad p^*(K_X+\Delta).\xi = 0$

but $K_Y + \Delta_Y$ is nef,

$q^*(K_Y+\Delta_Y).\xi = (K_Y+\Delta_Y) q_*\xi \geq 0$ $\quad \in \overline{NE}(Y)$

by assumption that $(Y, \Delta_Y) = $ WLCM

Negativity Lemma $\Rightarrow$ B effective ∎

<u>Cor</u> the singularities of WLCM are in the same class as those of (X, Δ)

<u>Def</u> a log minimal model (Y, Δ_Y) is <u>good</u> if $K_Y + \Delta_Y$ is f-semiample

Abundance = all minimal models are good

LMMP conj

let $f:(X,\Delta)\to B$ dlt pair proj $/B$

for all sequences

there are a sequence of $(K_X+\Delta)$-flips

+ divisorial extremal contractions s.t.

composition ϕ either

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr] & & Y \arrow[dl] \\ & B & \end{tikzcd}$$

a) is a good minimal $\Longleftrightarrow$ $K_X+\Delta$ pseudoeff

or

b) is a mori Fiber space

$\Longleftrightarrow$ $K_X+\Delta$ not pseudo effective

Thm (uniqueness of canonical models)

$f:(X,\Delta)\to B$ as before

$(Y,\Delta_Y)=LCM(X,\Delta/_B)$, then

$$Y=\mathrm{Proj}_B \bigoplus_{m\geq 0} f_*\mathcal{O}_X(mK_X+\lfloor m\Delta\rfloor)$$

Proof

reduced

q- exceptionals

$\Downarrow$

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

$$K_W + \underbrace{p_*^{-1}\Delta + \sum E_i}_{\Delta_W} = p^*(K_X+\Delta) + \text{effective} + p\text{-exc}$$

$$K_W + \Delta_W = q^*(K_Y+\Delta_Y) + \text{effective } q\text{-exc}$$

by def of LCM

by the eqn's + negativity lemma

$$p_* \mathcal{O}_W(mK_W + \lfloor m\Delta_W \rfloor) = \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor)$$

$$q_* \mathcal{O}_W(mK_W + \lfloor m\Delta_W \rfloor) = \mathcal{O}_Y(mK_Y + \lfloor m\Delta_Y \rfloor)$$

$$Y = \mathrm{Proj}_B \bigoplus_{m\geq 0} \underbrace{g_* \mathcal{O}_Y(mK_Y + \lfloor m\Delta_Y \rfloor)}$$

b/c $K_Y + \Delta_Y$ is g-ample

$$\begin{aligned} &= g_* q_* \mathcal{O}_W(mK_W + \lfloor m\Delta_W \rfloor) \\ &= f_* p_* \mathcal{O}_W(mK_W + \lfloor m\Delta_W \rfloor) \\ &= f_* \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor) \end{aligned}$$

B

__Rmk__ $LCM(X, \Delta/B)$ is unique if it exists

exists $\Leftrightarrow$ $\bigoplus_{m\geq 0} f_* \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor)$ f.g.

Cor $\varphi: X' \to X$ s.t. (X', Δ') lc

$K_{X'} + \Delta' = \varphi^*(K_X + \Delta) +$ eff φ-exc

$\Longrightarrow$ $LCM(X', \Delta'/B) = LCM(X, \Delta/B)$

Prop let $f: X \to Y$ be a $(K_X+\Delta)$-flipping contraction of an lc pair (X, Δ) then

1) the flip $f^+: X^+ \to Y$, if it exists, is the $LCM(X, \Delta/Y)$

2) the flip exists $\iff$ $\bigoplus_{m\geq 0} f_* \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor)$ f.g.

3) flips are unique

Proof

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

$\Delta^+ = \phi_* \Delta$

ϕ extracts no divisors

& has no exceptionals

$K_{X^+} + \Delta^+$ is f^+-ample by def of a flip

$\Longrightarrow$ $X^+ = LCM(X, \Delta/Y)$ by def

Inversion of adjunction

recall if $S \subseteq X$ cartier divisor

$(K_X + S)|_S = K_S \leftarrow$ $R: \omega_X(S) \to \omega_S$

$0 \to \omega_X \to \omega_X(S) \xrightarrow{R} \omega_S \to 0$

smooth case

$R\left(f \frac{dx_0}{x_0} \wedge \dots \wedge dx_n\right)$

$= f(0, x_1, \dots, x_n)\, dx_1 \wedge \dots dx_n$

$(K_X + S + \Delta)|_S + K_S + \Delta|_S$

$(X, S + \Delta) \longleftrightarrow (S, \Delta|_S)$

$(X, \Delta + S)$

Different: S normal integral weil divisor

$Z \subseteq S \subseteq X$ s.t. $X \setminus Z \supseteq S \setminus Z$
smooth cartier

$\Delta \cap S \subseteq Z$

S, Δ share no components

I) $R_{S\setminus Z} : \omega_X(S)|_{S\setminus Z} \xrightarrow{\sim} \omega_{S\setminus Z}$

$m(K_X + S + \Delta)$
Cartier for some $m > 0$

$$R^m_{S\setminus Z} : \omega_X^{[m]}(mS + m\Delta)|_{S\setminus Z} \xrightarrow{\sim} \omega_{S\setminus Z}^{[m]}$$

since S normal, $S^0 \subseteq S$ s.t.
everything is a line bundle on S^0
& $\operatorname{codim}(S \setminus S^0) \geq 2$
extend to some isomorphism

$$R^m_{S^0} : \omega_X^{[m]}(mS + m\Delta)|_{S^0} \xrightarrow{\sim} \omega_{S^0}^{[m]}(\Delta_{S^0})$$

Δ_{S^0} Cartier

$$\Longrightarrow \quad R^m_S : \omega_X^{[m]}(mS + m\Delta)|_S \xrightarrow{\sim} \omega_S^{[m]}(\Delta_S)$$

$\Delta_S = i_* \Delta_{S^0} \qquad i : S^0 \hookrightarrow S$

$$\operatorname{Diff}_S(\Delta) := \frac{\Delta_S}{m} \in \operatorname{WDiv}_{\mathbb{Q}}(X)$$

II) $(K_X + S + \Delta)|_S \sim_{\mathbb{Q}} K_S + \operatorname{Diff}_S(\Delta)$

III) $f : Y \to (X, S+\Delta)$ log resolution

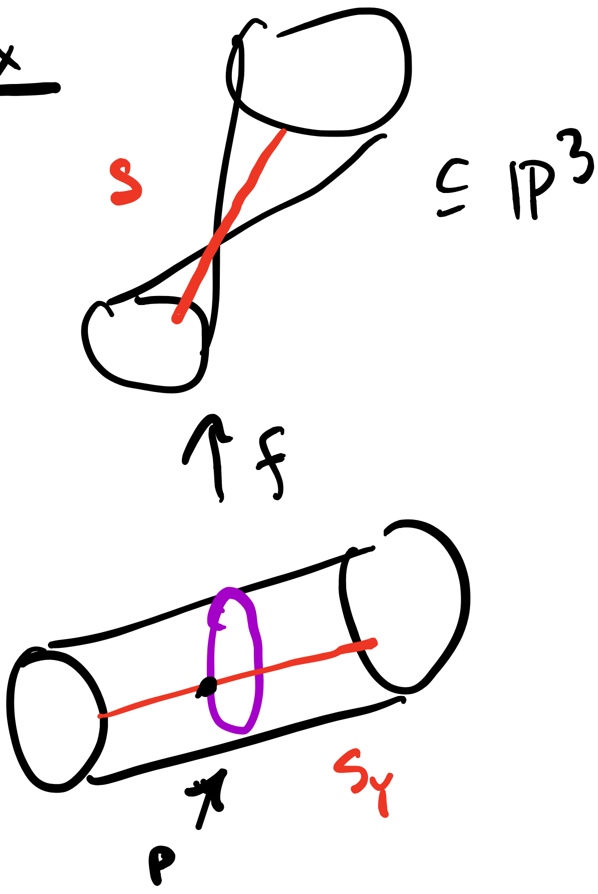
$$K_Y + S_Y + \Delta_Y = f^*(K_X + S + \Delta), \quad f_* S_Y = S$$

define S_Y, Δ_Y $\qquad f_* \Delta_Y = \Delta$

$$\Delta_Y|_{S_Y} = \mathrm{Diff}_{S_Y}(\Delta_Y)$$

$$\left(\begin{array}{l} \mathrm{Diff}_S(\Delta) = f_* \Delta_Y|_{S_Y} \\ f|_{S_Y}^*\left(K_S + \mathrm{Diff}_S(\Delta)\right) = K_{S_Y} + \overbrace{\mathrm{Diff}_{S_Y}(\Delta_Y)}^{\Delta_Y|_{S_Y}} \end{array} \right)$$

Ex



cone over a conic

$$f^* K_X = K_Y$$

$$f^* S \doteq S_Y + \frac{1}{2} E$$

$$\Delta_Y = \frac{1}{2} E$$

$$\Rightarrow \quad \mathrm{Diff}_{S_Y}(\Delta_Y) = \frac{1}{2} P$$

$$\mathrm{Diff}_S(0) = \frac{1}{2} \text{ (cone point)}$$