Inversion of adjunction

Recall $(X, S+\Delta)$ is a log pair

no components in common

S integral, normal divisor

there is a unique divisor $\mathrm{Diff}_S(\Delta)$ s.t.

1) $(K_X+S+\Delta)|_S \sim_{\mathbb{Q}} K_S + \mathrm{Diff}_S(\Delta)$

2) $f : (Y, S_Y+\Delta_Y) \to (X, S+\Delta)$

   s.t. $f_* S_Y = S$ $\quad f_*\Delta_Y = \Delta$

   $f^*(K_X+S+\Delta) = K_Y+S_Y+\Delta_Y$ crepant

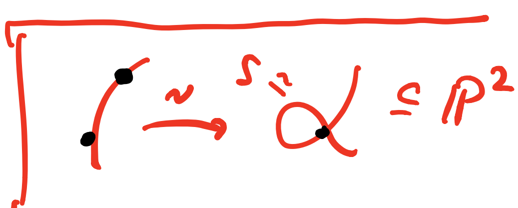
   $f_*(\Delta_Y|_{S_Y}) = \mathrm{Diff}_S(\Delta)$

   $f_S^*(K_S+\mathrm{Diff}_S(\Delta)) = K_{S_Y} + \Delta_Y|_{S_Y}$

3) if Δ effective, then $\mathrm{Diff}_S(\Delta)$ is effective

4) generalize to the case S is not normal: $\nu : S^n \to S$ the normalization

   Correct by $K_{S^n/S}$



$\nu^*(K_{\mathbb{P}^2}+S)$

$= K_{S^n} + 2$ points

Prop let (X,Δ) be a dlt pair, then TFAE

1) (X,Δ) plt 2) $\lfloor\Delta\rfloor$ normal 3) $\lfloor\Delta\rfloor$ is a disjoint union of irred comp

Pf 1) $\Rightarrow$ 2)

$f: Y \to X$ a log resolution

$$\boxed{K_Y + \Delta_Y = f^*(K_X+\Delta) + A}$$

by plt

$\Delta_Y = f_*^{-1}\Delta$ by plt, $\lceil A\rceil = E$ effective f-exc

$S = \lfloor\Delta\rfloor$ $S_Y = \lfloor\Delta_Y\rfloor$

$f_* S_Y = S$

$$0 \to \mathcal{O}_Y(-S_Y+E) \to \mathcal{O}_Y(E) \to \mathcal{O}_{S_Y}(E|_{S_Y}) \to 0$$

fractional part

$$-S_Y + E = -\lfloor\Delta_Y\rfloor + \lceil A\rceil \equiv_f K_Y + \{\Delta_Y\} + \lceil A\rceil - A$$

$(Y, \{\Delta_Y\} + \lceil A\rceil - A)$ is klt

by GR vanishing, $R^1 f_* \mathcal{O}_Y(-S_Y+E) = 0$

$$\Rightarrow f_*\mathcal{O}_Y(E) \twoheadrightarrow f_*\mathcal{O}_{S_Y}(E|_{S_Y}) \to 0$$

$\parallel$

$\mathcal{O}_X$ b/c E effective exc

$$\mathcal{O}_X \to \mathcal{O}_S \to f_*\mathcal{O}_{S_Y} \hookrightarrow f_*\mathcal{O}_{S_Y}(E|_{S_Y})$$

$\Rightarrow$ $\mathcal{O}_S \twoheadrightarrow f_*\mathcal{O}_{S_Y}$ surjective

$\Rightarrow$ isomorphism & $f_*\mathcal{O}_{S_Y} = \mathcal{O}_{S^n}$

2) $\Rightarrow$ 3) by def 3) $\Rightarrow$ 1) by computing disc

$\boxed{}$

<u>Cor</u> suppose (X, Δ) is dlt

$S \subseteq \lfloor \Delta \rfloor$ irr component, write $\Delta = S + \Delta_1$

then S is normal.

<u>Pf</u> $S + (1-\varepsilon)\Delta_1 + \varepsilon D$ is dlt

not $\mathbb{Q}$-cartier

<u>Lemma</u> (2.43)

$\exists$ $D \equiv \varepsilon\Delta_1 + H$

ample

so $(X, S + (1-\varepsilon)\Delta_1 + \varepsilon D)$ is dlt

but $\lfloor S + (1-\varepsilon)\Delta_1 + \varepsilon D \rfloor = S$

$\Rightarrow$ S is normal by the previous prop.

Prop let $(X, S+\Delta)$ log pair, S integral & normal. Then for every divisor E_S lying over S there exists a divisor E lying over X s.t.

$$a(E_S, S, \mathrm{Diff}_S(\Delta)) = a(E, X, S+\Delta)$$

Moreover

$$\text{total disc}(S, \mathrm{Diff}_S(\Delta)) \geq \text{discrep}(\text{center} \subseteq S, X, S+\Delta)$$
$$\geq \text{discrep}(\text{centers}, X, S+\Delta)$$

Pf

$f : (Y, S_Y + \Delta_Y) \to (X, S+\Delta)$ crepant log resolution

$$f^*(K_S + \mathrm{Diff}_S(\Delta)) = K_{S_Y} + (\Delta_Y)|_{S_Y}$$

Suppose that S_Y is disjoint from $f_*^{-1}(\Delta)$, then all divisors we're dealing with are f-exc

take a sequence of blowups of S s.t. E_S appears as a divisor on this blowup, blowup the same sequence of subs of S but in X, this gives a divisor $E \subseteq Y$ s.t. $E|_{S_Y} = E_{S_Y}$

s.t. $\text{Center}_{S_Y}(E_S) = E_{S_Y}$

discrep $=$ $-$ coefficient for a divisor appearing on Y

$$(K_Y + S_Y + \Delta_Y)|_{S_Y} = K_{S_Y} + (\Delta_Y)|_{S_Y}$$

$\square$

<u>Cor</u> (Easy adjunction)

1) if $(X, S+\Delta)$ is plt in a nbhd of S

$(S, \text{Diff}_S(\Delta))$ is klt

2) if $(X, S+\Delta)$ is lc in a nbhd of S

$(S, \text{Diff}_S(\Delta))$ is lc

<u>Thm</u> (Inversion of adjunction) (Shokurov, Kollár, Kawakita)

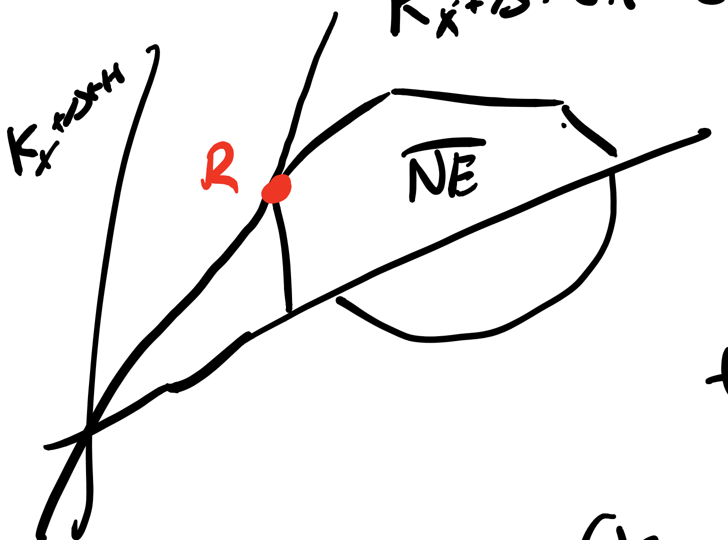
The inequalities of discrep above are $=$, therefore the statements of the cor are $\Leftrightarrow$

<u>MMP with scaling</u>

(X, Δ) klt pair, H be some ample

$K_X + \Delta + H$ nef

$K_X + \Delta + tH = 0$ for $t \le 1$



t_1 = nef threshold

$(K_X + \Delta + t_1 H) . R = 0$
for some extremal ray
R

$(K_X + \Delta)$-negative extremal ray
if $t_1 > 0$

$\phi_R : X \to Z$ the $(K_X + \Delta)$-extremal contraction

$(Z, \Delta_Z + t_1 H_Z) = WLCM(X, \Delta + t_1 H/B)$

if ϕ_R is small, X_1 = flip

if ϕ_R is divisorial, $X_1 = Z$

if ϕ_R is a Mori fiber space

$(X_1, \Delta_1 + t_1 H_1)$ WLCM so $K_{X_1} + \Delta_1 + t_1 H_1$

$t_2 < t_1$ $\qquad$ $t_2 =$ nef threshold

is nef

$LCM(X_1, \Delta_1 + t_2 H_1 / B) =$ image of extremal contraction of R_1

$(K_{X_1} + \Delta_1 + t_2 H_1) . R_1$

technical point: need some bigness of Δ to guarantee that R_1 is $(K_{X_1} + \Delta_1)$-neg extremal ray

$$X_2 = WLCM(X_1, \Delta_1 + t_2 H_1 / B)$$
$$= WLCM(X, \Delta + t_2 H / B)$$
$$\vdots$$

$$0 < \dots < t_2 < t_1 < 1$$

each step of MMP occurs at coefficient t_i, and produces

$WLCM(X, \Delta + t_i H / B)$

Termination of MMP with scaling $\iff$ Finiteness of WLCM for $(X, \Delta + tH)$ as $t \in [0, 1]$

pl - flips (pre-limiting, Shokurov)

Def let (X, Δ) plt pair, $S = \lfloor \Delta \rfloor$ is integral. a pl-flipping contraction is a $(K_X + \Delta)$-flipping contraction $\phi : X \to Z$ s.t. $-S$ is ϕ-ample.

---

suppose Z is affine,

existence of flips = $R(X, K_X + \Delta)$

$\Delta = S + B \qquad B = \{\Delta\}$

$$0 \to H^0(X, m(K_X + B)) \to H^0(X, m(K_X + S + B))$$
$$\to H^0(S, m(K_S + \mathrm{Diff}_S(B)))$$

$$\mathrm{Res}: R(X, K_X + \Delta) \longrightarrow R(S, K_S + \mathrm{Diff}_S(B))$$

Shokurov suffices to prove

image of Res is f.g.

$R_S(X, K_X + \Delta)$

Hacon - McKernan MMP dim $n-1$
$\Longrightarrow$ existence of pl-flips

$$MMP_{n-1} \Rightarrow \text{pl-flips}_n$$

Shokurov MMP_{n-1} + pl-$\text{flips}_n \Longrightarrow \text{flips}_n$

Existence of minimal models for **varieties of log general type**

BCHM = Birkar - Cascini - Hacon - McKernan

Thm I $f : (X, \Delta) \to B$ $\mathbb{Q}$-factorial klt pair projective $/B$, suppose Δ is f-big. Then any MMP with scaling terminates.

Thm II let $f : (X, \Delta) \to B$ as above, suppose either Δ is big and $K_X + \Delta$ is f-pseudo effective

OR $K_X + \Delta$ is f-big, then

1) (X, Δ) has a good LTM $(X, \Delta/B)$

2) if $K_X + \Delta$ is f-big, then (X, Δ) has a $LCM(X, \Delta/B)$

3) $R(\pi, K_X + \Delta) = \bigoplus_{m \geq 0} f_* \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor)$ is f.g.

<u>Pf sketch</u>

if $K_X + \Delta$ is f-big

$K_X + \Delta \sim_{f, \mathbb{Q}} D \geq 0$

$\Delta' = \Delta + \varepsilon D$, Δ' is big

(X, Δ') is klt

$K_X + \Delta' \equiv (1+\varepsilon)(K_X + \Delta)$

$(K_X + \Delta)$-MMP $= K_X + \Delta'$ MMP

but now, Δ' is big so by thm I, MMP with scaling terminates

$$\begin{array}{ccc} X & \dashrightarrow & LTM(X, \Delta/B) \\ & \searrow_{B} & \swarrow \end{array}$$

b/c $K_X + \Delta$ is pseudoeffective

2) + 3) in the case $K_X + \Delta$ is big is just the bpf

3) in the case Δ is big

$K_X + \Delta \equiv K_X + A + B$ $\quad$ A ample

(X, B) klt

$\overbrace{(K_X + \Delta)}^{D} - (K_X + B) = \text{Ample}$

in the bpf theorem, $\Longrightarrow$ bpf

$\Longrightarrow$ models are good + finite generation

$\square$