BCHM - Existence of minimal models for varieties of log general type

Thm I $f:(X,\Delta)\to B$ is a $\mathbb{Q}$-factorial klt pair projective$/B$. Suppose Δ $K_X+\Delta$ is f-big. Then any MMP over B with scaling terminates.

Thm II In the same setup, suppose either a) Δ is f-big & $K_X+\Delta$ is f-pseudo effective, or

b) $K_X+\Delta$ is f-big, then

1) (X,Δ) has a $LTM(X,\Delta/B)$

2) if $K_X+\Delta$ is f-big, then $LCM(X,\Delta/B)$ exists

3) $R(f, K_X+\Delta) := \bigoplus_{m\geq 0} f_* \mathcal{O}_X(mK_X+\lfloor m\Delta\rfloor)$ is finitely generated

4) any $WLCM(X,\Delta/B)$ is good

Cor 1 let (X, Δ) is a projective klt pair. Then $R(K_X+\Delta)$ is finitely generated.

Rmk no assumptions on Δ or $K_X+\Delta$ big.

Pf Thm II $\Rightarrow$ in the case that $K_X+\Delta$ is big

Fujino - Mori

take Iitaka fibration

$$\begin{tikzcd} X \arrow[r, dashed, "\varphi_{|dK_X+d\Delta|}"] & Z \\ Y \arrow[u, "f"] \arrow[ur, "\phi"'] & \end{tikzcd}$$

Pick divisors on Y s.t.

(Y, Δ_Y) s.t.
1) klt
2) $f^*(K_X+\Delta) = K_Y + \Delta_Y$
3) $f_* \Delta_Y = \Delta$
4) ...

$$R(K_X+\Delta) = R(K_Y+\Delta_Y)$$

$\dim Z = \kappa(K_X+\Delta)$

ϕ is a "$(K+\Delta)$-trivial fibration"

Fujino-Mori canonical bundle formula

$$K_Y + \Delta_Y \sim_{\mathbb{Q}} f^*(K_Z + L + B) + B'$$

L - measuring the way fibers of f vary in moduli

B - measuring singularities of fibers

$$\Longrightarrow \quad R(K_Y + \Delta_Y) = R(K_Z + L + B) \leftarrow \text{f. g. by BCHM}$$

$$\dim Z = \kappa(K_X + \Delta) = \kappa(K_Y + \Delta_Y) = \kappa(K_Z + L + B)$$

$\boxtimes$

Cor 2 $f : (X, \Delta) \to B$ as above.

Suppose $K_X + \Delta$ is <u>not</u> pseudoeff.

Then there exists an $(K_X + \Delta)$-MMP relative to B that terminates in a Mori fiber space $/B$.

Pf pick A an f-ample s.t.

$K_X + \Delta + A$ is f-ample, & s.t.

$(X, \Delta + A)$ is klt.

$K_X + \Delta + \varepsilon A$ for $0 < \varepsilon << 1$

$\uparrow$

not pseudo effective

$\Delta' := \Delta + \varepsilon A$ is big

run a $K_X + \Delta'$ MMP with scaling by A

terminates by thm I

since $K_X + \Delta'$ <u>not</u> pseudoeffective it terminates in a Mfs

<u>but</u> $K_X + \Delta'$ MMP is also a $K_X + \Delta$ MMP ⊛

<u>Rmk</u> Case thats open is when $K_X + \Delta$ is pseudoeffective but neither Δ nor $K_X + \Delta$ are big.

<u>Cor</u> (X, Δ) klt pair, $f: X \to Z$ is a $K_X + \Delta$ flipping contraction. Then the flip of f exists.

<u>Pf</u> If the flip exists, its equal to $LCM(X, \Delta / Z)$

$-(K_X + \Delta)$ is f-ample

$\sim_{\mathbb{Q},f} D \geq 0$

(X, Δ') is klt

$\Delta' = \Delta + \varepsilon D$

Δ' is f-big

$$K_X + \Delta' \equiv (1-\varepsilon)(K_X + \Delta)$$

$$LCM(X, \Delta/Z) = LCM(X, \Delta'/Z)$$ exists by Thm II

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outline of main induction

$f: (X, \Delta) \to B$
klt proj $/B$

A Existence of pl-flips

B Special finiteness of models

$\Delta = S + A + \Delta'$ — S integral weil divisor, A ample

as $\Delta' \geq 0$ varies over all divisors s.t. (X, Δ) as above, there are finitely many birational models of X s.t. any $LTM(X, \Delta/B)$ is isomorphic to one of these finitely many in a nbhd of S

C Existence of LTM over B when Δ is big $/B$

D $K_X + \Delta$ is pseudoeffective,
Δ big $\implies$ $K_X + \Delta \sim_{\mathbb{R}, f} D \geq 0$

E finiteness of models:
$\Delta = A + \Delta'$ $\quad$ A ample
then as Δ' varies over all possible effective divisors s.t. (X, Δ) is klt, there are finitely many $WLCM(X, \Delta / B)$

F finite generation + Zariski decomposition

$(\text{finite generation})_{n-1} \implies (\text{pl-flips})_n$

$(\text{special finiteness})_n + (\text{pl-flips})_n \implies LTM_n$

$(\text{finiteness})_{n-1} \implies (\text{special finiteness})_n$

$(\text{non-vanishing})_{n-1} + (\text{special-fin})_n + LTM_n$
$\implies (\text{non-vanishing})_n$

$LTM_n + (\text{non-vanishing})_n \implies (\text{finiteness})_n$

$LTM_n + (\text{non-vanishing})_n + (\text{finiteness})_n \implies \left(\begin{smallmatrix}\text{finite}\\ \text{gen}\end{smallmatrix}\right)_n$

X is always reduced

The case of non-normal varieties

Why?

1) apply induction arguments to the log canonical case, then need to work with non-normal S for the induction

2) to construct proper of moduli spaces of higher dimensional varieties

Divisor theory

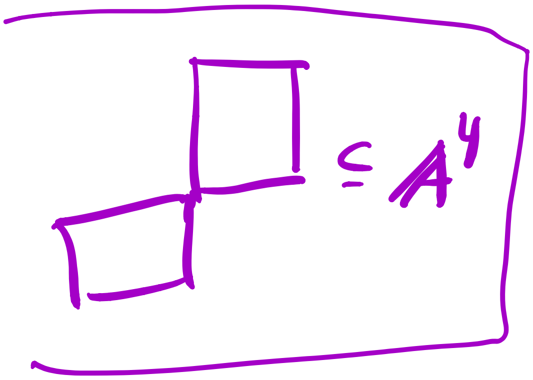
consider D such that the generic point η of D lies in the smooth locus of X

$\mathcal{O}_{X,\eta} \leftarrow$ dVR

Weil divisors + linear equivalence generalize as is

Need to assume that X is S_2 ← Serre's condition S_2

S_n: for each $x \in X$,
$\operatorname{depth}(\mathcal{O}_{X,x}) \geq \min\{n, \dim \mathcal{O}_{X,x}\}$



Canonical sheaf

G1 - gorenstein in codim 1

there exists an open set $U \overset{j}{\subseteq} X$ open

s.t. 1) $X \setminus U$ has codim ≥ 2

2) U is gorenstein, i.e. it has a canonical line bundle ω_U

If X is reduced, G1, S2

$$\omega_X = j_* \omega_U = \mathcal{O}_X(K_X)$$

a weil divisor class as above

$\nu : X^n \to X$ the normalization

conductor: $0 \to \mathcal{O}_X \to \nu_* \mathcal{O}_{X^n} \to \mathcal{F} \to 0$

$$\mathrm{cond}(X) = \underset{\mathcal{O}_X}{\mathrm{ann}}(\mathcal{F}) \subseteq \mathcal{O}_X \subseteq \nu_* \mathcal{O}_{X^n}$$

$$\| $$

$$\{a \in \mathcal{O}_X \mid af \in \mathcal{O}_X \text{ for all } f \in \mathcal{O}_{X^n}\}$$

$$D = V(\mathrm{cond}(X)) \subseteq X^n$$

$$\nu^* K_X = K_{X^n} + D$$

Grothendieck duality on the gorenstein locus

<u>Ex</u> X surface, $D \subseteq X$ reduced divisor s.t. (X, D) is log canonical

$\Longrightarrow$ D has at worst nodal singularities

So in general, (X, D) $\dim X \geq 2$

D reduced, (X, D) log canonical,

$\Longrightarrow$ D is nodal in codim 1

From the pov of MMP,

X s.t. 1) reduced

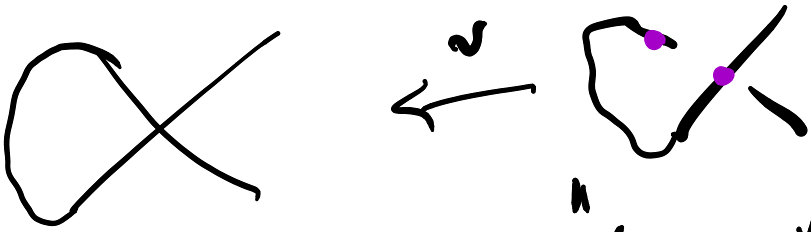
2) nodal in codim 1 ($\Rightarrow$ G1)

3) S_2

Compute conductor

$$\frac{k[x,y]}{(xy)} \hookrightarrow k[x]\times k[y]$$

normalization ν_*



C $\quad$ $\mathbb{P}^1$

$$\nu^* K_C = K_{\mathbb{P}^1} + P_1 + P_2$$

in the nodal case, we call

$D = V(\text{cond})$ ← double locus

← reduced divisor on X^n

(X, Δ) $\qquad K_X + \Delta \rightsquigarrow K_{X^n} + D + \Delta^n$

Def (X,Δ) is semi-log canonical

if 1) X is nodal in codim 1 & S_2

2) $K_X + \Delta$ $\mathbb{Q}$-Cartier

3) $\nu^*(K_X+\Delta) = K_{X^n} + \Delta^n + D$

$(X^n, \Delta^n + D)$ is log canonical