Stable pairs and their moduli

semi-log canonical singularities
(slc)

(X, Δ) 1) reduced, S_2, nodal in codim 1

demi-normal

$$\begin{tikzcd} X^n \arrow[r, "\nu"] & X \\ D^n \arrow[u, hook] \arrow[r, "\text{gen } 2:1"] & D = V(\text{cond}) \arrow[u, hook] \end{tikzcd}$$

norm

$$\nu^*(K_X + \Delta) = K_{X^n} + \Delta^n + D^n$$

2) $(X^n, \Delta^n + D^n)$ is log canonical

3) $K_X + \Delta$ is $\mathbb{Q}$-Cartier

Stable pairs = semi-log canonical model
(slc)

(X, Δ) is a stable pair if

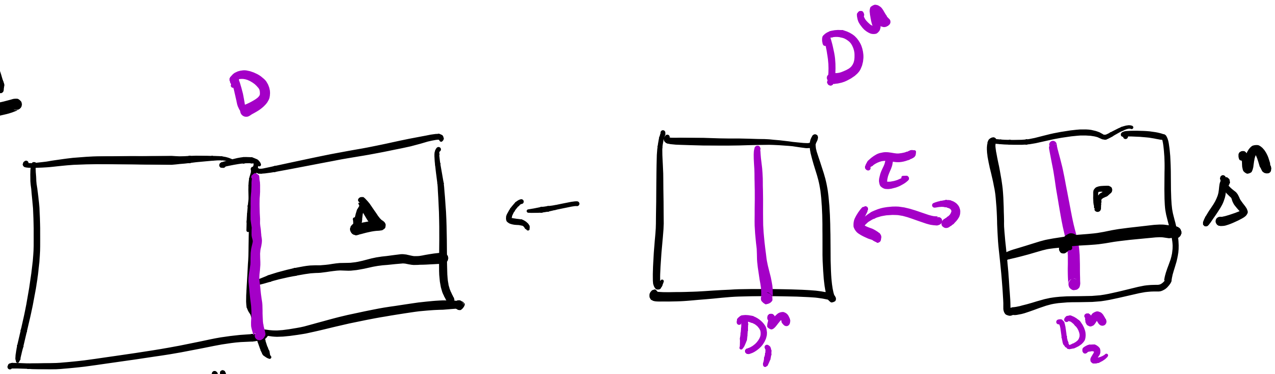
1) it has slc singularities

2) $K_X + \Delta$ is an ample $\mathbb{Q}$-Cartier divisor

$$\begin{tikzcd} D^n \arrow[loop left, dashed] \arrow[r, hook] \arrow[d, "2:1"'] & X^n \arrow[d] \\ D \arrow[r, hook] & X \end{tikzcd}$$

this is a pushout in schemes

Ex 1



$\| \quad (X,\Delta)$

$(X_1^n, D^n + \Delta^n)$ is lc

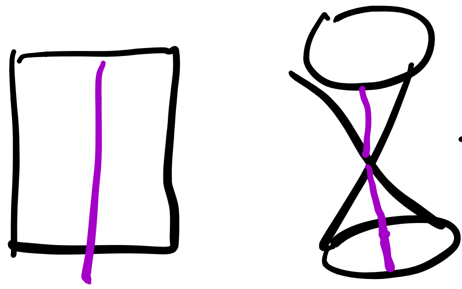
$$\nu^*(K_X+\Delta)|_{D_1^n} = (K_{X^n} + D^n + \Delta^n)|_{D_1^n} = K_{D_1^n} + 0 = -2$$

$$\nu^*(K_X+\Delta)|_{D_2^n} = (K_{X^n} + D^n + \Delta^n)|_{D_2^n} = K_{D_2^n} + p = -1$$

necessary cond for $\mathbb{Q}$-Cartier:

$\mathrm{Diff}_{D^n}(\Delta^n)$ is τ-invariant

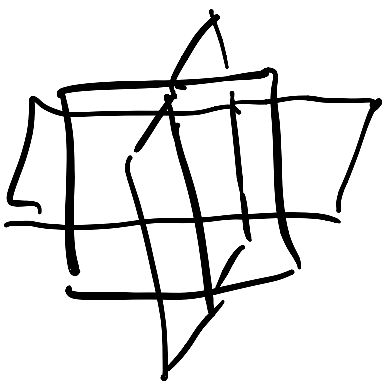
2)



$= (X^n, D^n) \overset{\text{glue}}{\rightsquigarrow} X$

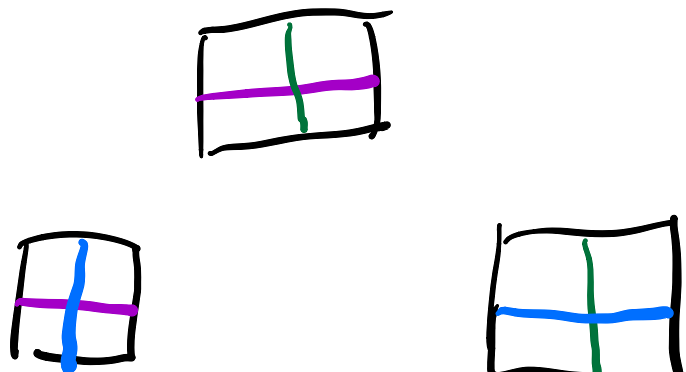
K_X not $\mathbb{Q}$-Cartier

3)



$xyz=0 \cap \mathbb{A}^3$

$= X \xleftarrow{\sim}$



here τ is not defined at the origin

but: $\tau : \overline{D^n} \to \overline{D^n}$

$\overline{D^n}$ ↗ normalization of D^n

$$\left[(X, \Delta^n + D^n),\ \underset{\text{involution}}{\tau : \overline{D^n} \to \overline{D^n}},\ \tau \text{ preserves } \mathrm{Diff}_{\overline{D^n}}(\Delta^n) \right]$$

not a closed embedding

$$\begin{tikzcd}
\overline{D^n} \arrow[r] \arrow[d] & X^n \arrow[d, dashed] \\
\overline{D^n}/\tau \arrow[r, dashed] & X
\end{tikzcd}$$

want to take a ~~push~~out

doesn't exist

Kollár

1) such quotients exist in the log canonical

2) $K_X + \Delta$ is $\mathbb{Q}$-Cartier $\iff$ $\mathrm{Diff}_{\overline{D^n}}(\Delta^n)$ is τ-invariant

Thm (Kollár)

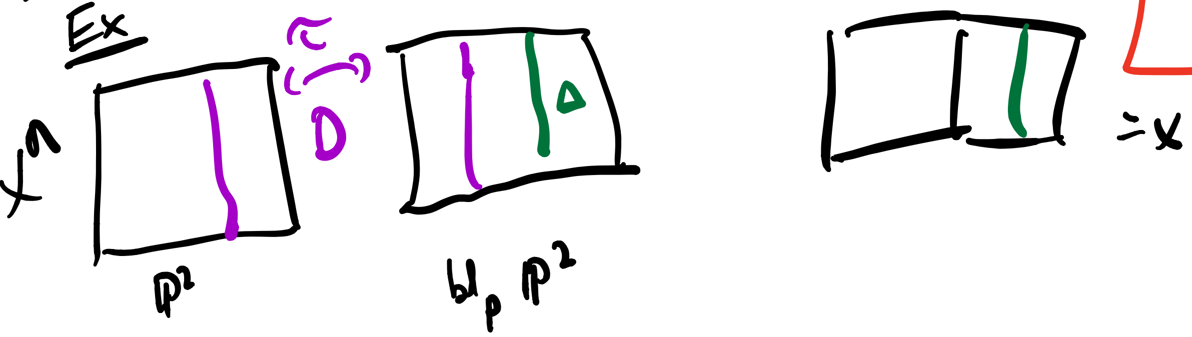
$$\left\{ \begin{array}{c} \text{semi-log canonical} \\ \text{stable pairs} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{triples of} \\ (X, \Delta^n + D^n) \text{ s.t. } K_{X^n} + \Delta^n + D^n \text{ ample} \\ \tau : \overline{D^n} \to \overline{D^n} \text{ different preserving} \end{array} \right\}$$

Strategy for MMP

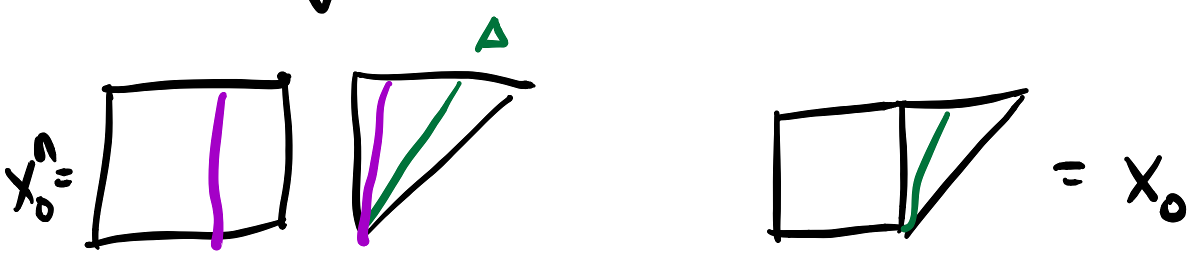
- input (X, Δ) slc
- normalize $(X^n, \Delta^n + D^n)$
- run mmp ↑ here to get $LCM(X^n, \Delta^n + D^n)$
- re-glue to get the stable model of (X, Δ)

does not work in general

Ex

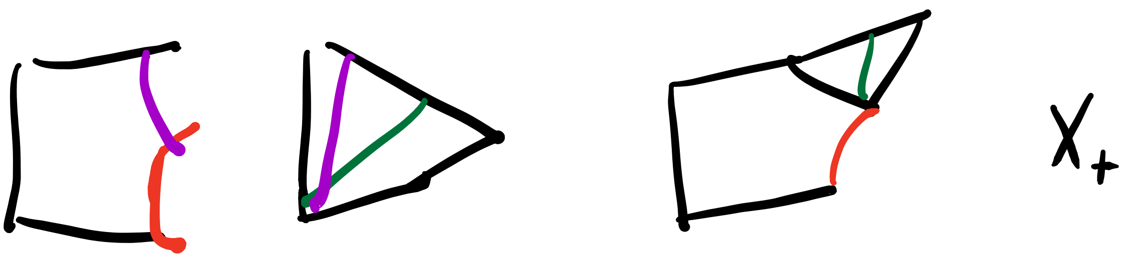


↓ lc model



K_{X_0} not $\mathbb{Q}$-Cartier

hidden flip



K_{X_+} $\mathbb{Q}$-Cartier

Kollár examples of slc surfaces s.t. $R(K_X + \Delta)$ not f.g.

Lemma $(X, S+\Delta)$ is log canonical s.t. $\lfloor S+\Delta \rfloor = S$. Suppose S is S_2, then $(S, \mathrm{Diff}_S(\Delta))$ is slc

Alexeev

if S is Cartier then its automatically S_2

Moduli of higher dimensional varieties

{collection of varieties} $= \mathcal{C}$

Moduli space: space $\mathcal{M}$ with a morphism π s.t.

$$\begin{array}{c} \mathcal{X} \\ \downarrow \pi \\ \mathcal{M} \end{array}$$

1) π is a flat family of varieties in $\mathcal{C}$

2) $\mathcal{M}(\mathbb{C}) \longrightarrow \mathcal{C}$ $\quad x \in \mathcal{M}(\mathbb{C}) \mapsto \mathcal{X}_x \in \mathcal{C}$

3) $(\mathcal{M}, \pi)$ satisfies a universal property represents a functor

Checklist of wants for $\mathcal{M}$

1) separated
2) finite type (boundedness)
3) proper
4) projective (coarse moduli space)

1-dimensional example

$$\mathcal{M}_{g,n} = \left\{ (C, p_1, \dots, p_n) \;\middle|\; \begin{array}{l} C \text{ smooth proj genus } g \text{ curve}, \\ p_i \text{ marked points} \end{array} \right\}$$

$\bigcap$ $2g-2+n>0$ $(C, \sum p_i)$ is an LCM

$$\overline{\mathcal{M}}_{g,n} = \left\{ \text{———} \;\middle|\; \begin{array}{l} C \text{ is at worst nodal, proj genus } g \\ p_i \in \text{smooth points} \\ K_C + \sum p_i \text{ ample} \end{array} \right\}$$

'69 Deligne–Mumford

upshot $\overline{\mathcal{M}}_{g,n}$ parametrizes stable pairs = slc models in dimension 1

$\overline{\mathcal{M}}_{g,n}$ satisfies our checklist!

Want the same in higher dim

1) $\left\{\begin{array}{c}\text{moduli of smooth} \\ \text{projective} \\ \text{varieties}\end{array}\right\}$ not separated if dim ≥ 2 !

val criterion of separated:

if limits exist $\Rightarrow$ unique

MMP $\Downarrow$

analogue of $\mathcal{M}_{g,n}$

is the space of $LCM(X,\Delta)$

2) boundedness/finite type

HMX – if you fix the coefficient set of Δ, &

fix volume $\mathrm{vol}(K_X+\Delta)$

$\Rightarrow$ the $LCM(X,\Delta)$ form a bounded family

3) if you all semi-log canonical stable pairs, then the functor / moduli space is proper

Valuative criteria for proper

all limits exist & are unique

$$\begin{array}{ccccc} (X^0, \Delta^0) & \subseteq & X & \supseteq & X_0 \\ \downarrow & & \downarrow & & \\ C^0 & \subseteq & C = \operatorname{Spec} DVR & \ni & 0 \end{array}$$

n+1fold

family of LCM in dim n

(X^0, Δ^0) is n+1 dim relative $LCM(X^0, \Delta^0 / C^0)$

$$(X, \Delta + X_0) \underset{\text{apply}}{\overset{\text{resolve MMP}}{\rightsquigarrow}} LCM(X, \Delta + X_0 / C)$$

$$= (X', \Delta' + X_0')$$

by adjunction $\Rightarrow$ $(X_0', \operatorname{Diff}_{X_0'}(\Delta'))$ is semi-log canonical + stable

so upshot:

stable pairs form a proper moduli space

Kollár–Shepherd-Barron surfaces
Alexeev
BCHM, HX Existence of lc closures

4) Kollár – projectivity of complete moduli ~ '90

Kovács–Patakfalvi –
proved projectivity for moduli of stable pairs

<u>Thm</u> (many people)
Stable pairs of fixed volume & coefficient set form a proper moduli space w/ projective coarse space.

5) <u>moduli functor</u>

Families of stable pairs $\neq$ flat morphisms with fibers being stable pairs

Kollár - Moduli of varieties of general type