Last time

X smooth projective surface

$X = X_0 \to X_1 \to \cdots \to X_m$ minimal surface

Theorem: Suppose K_{X_m} is not nef then either

1) $X_m \xrightarrow{\varphi} C$ $\quad X_m \cong \mathbb{P}_C(\mathcal{E})$
$\mathbb{P}^1$-bundle

2) X_m is Fano
$-K_{X_m}$ ample

minimal surface + Fano $\Rightarrow \mathbb{P}^2$

Rmk (Exercise)

in this case

$H^0(X_m, dK_{X_m}) = 0$
$\forall \; d > 0$

$\Rightarrow \; \kappa(X) = -\infty$

Strategy: by assumption

$K_X \cdot C < 0$ for some C

Want to study these K_X negative curves & use them to construct a linear series

§1: Cone of Curves

$$NE(X) = \left\{ \sum a_i [C_i] \;\middle|\; \begin{array}{l} a_i \in \mathbb{R}_{>0} \\ C_i \text{ effective curve} \end{array} \right\}$$

$$\subseteq N_1(X)_{\mathbb{R}} = N^1(X)_{\mathbb{R}}$$

$$\overline{NE}(X) = \text{closure of } NE(X)$$

$\langle\ ,\ \rangle : V \otimes V \to \mathbb{R}$ — for us $V = N^1(X)$, $\langle\ ,\ \rangle$ = intersection

$K \subseteq V$ some convex cone

$K^* \subseteq V$ $\quad K^* = \{ x \in V \mid \langle x, y \rangle \geq 0 \ \forall\, y \in K \}$

Nef cone: $\mathrm{Nef}(X) = \overline{NE}(X)^*$

Ample cone:

$$\mathrm{Amp}(X) = \{ x \in N^1(X) \mid \langle x, y \rangle > 0 \ \forall\, y \in \overline{NE}(X) \}$$

Kleiman's criterion

$$= \text{interior of } \mathrm{Nef}(X)$$

Corollary

Rmk: $\langle\ ,\ \rangle : N^1(X)\otimes N_1(X)\to \mathbb{R}$

in higher dimension

Examples

1) $X = E\times E$ $\qquad$ E elliptic curve

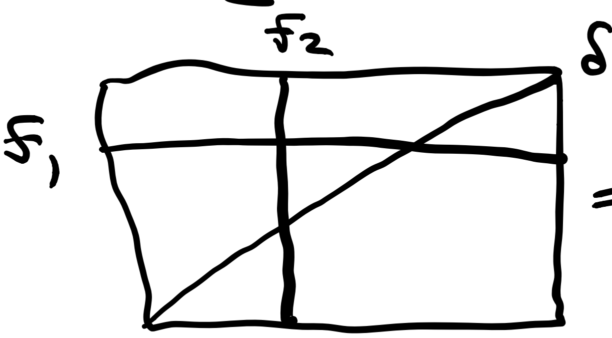
$\mathrm{End}(E) = \mathbb{Z}$

$$\mathrm{Pic}(C_1\times C_2) = \mathrm{Pic}(C_1)\times \mathrm{Pic}(C_2)\times \mathrm{Hom}(\mathrm{Jac}(C_1), \mathrm{Jac}(C_2))$$

$$\mathrm{Pic}(X) = (E\times\mathbb{Z})\times(E\times\mathbb{Z})\times \underbrace{\mathrm{End}(E)}_{\mathbb{Z}}$$

$N^1(X)_{\mathbb{Z}} = \mathbb{Z}^3$ $\qquad$ $N^1(X) = \mathbb{R}^3$

$= \mathrm{Span}(f_1, f_2, \delta)$



$= E\times E$

$f_1^2 = 0$ $\qquad$ $f_2^2 = 0$ $\qquad$ $\delta^2 = \deg N_{\Delta/X}$

$= \deg T_E$

$\Delta : E \hookrightarrow E\times E$ $\qquad$ $= \deg \mathcal{O}_E = 0$

$$f_1 \cdot f_2 = f_1 \cdot \delta = f_2 \cdot \delta = 1$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{"intersection matrix"}$$

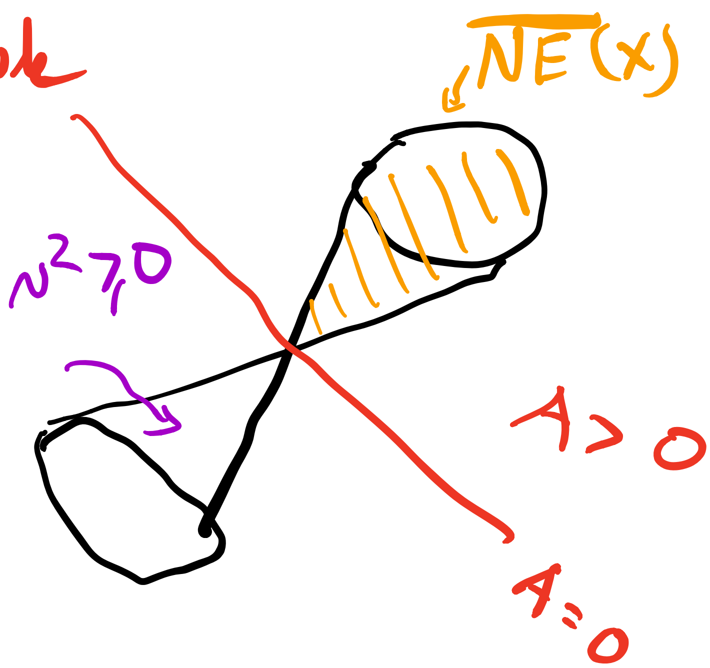
$v \in N^1(X)$ $\qquad v = (x, y, z)$

$\qquad\qquad\qquad\qquad xf_1 + yf_2 + z\delta$

$$v^2 = 2xy + 2xz + 2yz$$

Prop: $\overline{NE}(X) = \overbrace{\{v \mid v^2 \geq 0\} \cap \{A.v > 0\}}^{K:=}$

Fix A an ample



Rmk Here $NE(X) \neq \overline{NE}(X)$

Pf

$$\mathrm{Amp}(X) \subseteq NE(X) \underset{\text{claim}}{\subseteq} \mathrm{Nef}(X) = \overline{NE}(X) \subseteq K$$

Indeed, if $D, C \subseteq X$ we have a transitive group action of $E \times E$ on X

$\tau . D$ such that $\tau . D \cap C$ is finite
$$\tau \in E \times E$$

$$\Rightarrow \quad D . C = \tau D . C = \sum_{p \in \tau D \cap C} \mathrm{mult}_p(\tau D, C) \geq 0$$

<u>OTOH</u>

$\nu \in K^{int} \cap N^1(X)_{\mathbb{Q}} \Rightarrow \nu$ is effective

$m\nu = [D]$ $\quad \Omega_X = \Omega_{E \times E} = \mathcal{O}_X^{\oplus 2}$

$m > 0$ $\quad K_X = 0$

$$\chi(\mathcal{O}_X(D)) \overset{RR}{=} \frac{1}{2} D . (D - \cancel{K_X}^{\,0}) + \cancel{\chi(\mathcal{O}_X)}^{\,0}$$

$$\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X)$$

1 $\quad \overset{\text{Hodge Symmetries}}{||}$ $h^0(\Omega_X) = 2$ $\quad || \; h^0(\mathcal{O}_X) = 1$

$= 0$

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2} D^2 = \frac{\nu^2}{2} m^2 > 0$$

$||$

because $\nu \in$ interior of K

$$h^0(\mathcal{O}_X(D)) - h^1(\mathcal{O}_X(D)) + h^2(\mathcal{O}_X(D))$$

either

1) $h^0(\mathcal{O}_X(D)) > 0$ i.e. D effective

or

2) $h^2(\mathcal{O}_X(D)) > 0$

$$= h^0(\omega_X(-D)) = h^0(\mathcal{O}_X(-D))$$

$\Rightarrow$ $-D$ effective $\Rightarrow$ $(-D).A > 0$

$$\underline{\text{contradiction}} \qquad \overset{\shortparallel}{-(D.A) < 0}$$

$\Rightarrow$ $D = mv$ is effective

so $K = \text{closure}(K^{int} \cap N^1(X)_{\mathbb{Q}})$

$$\subseteq \overline{NE}(X)$$

$\Rightarrow$ $K = \overline{NE}(X)$ $\square$

Example 2

$$X = \mathbb{P}_C(\mathcal{E}) \xrightarrow{\pi} C \qquad \text{ruled surface}$$

$\mathcal{O}_X(1)$ universal quotient

$\mathcal{E}$ has even degree $2k$

replace $\mathcal{E} \otimes \mathcal{L}$ $\quad \deg \mathcal{L} = -k$

$\deg(\mathcal{E} \otimes \mathcal{L}) = 0$ $\quad + \quad$ $\mathbb{P}_C(\mathcal{E} \otimes \mathcal{L}) = \mathbb{P}_C(\mathcal{E})$

so wlog, $\deg \mathcal{E} = 0$

Projective bundle formula

$$\mathrm{Pic}(X) = \pi^* \mathrm{Pic}(C) \oplus \mathbb{Z}\, \mathcal{O}_X(1)$$

$\xi = [\mathcal{O}_X(1)]$ numerical class

$$\xi^2 + \underset{0}{\cancel{c_1(\mathcal{E})}}\,\xi + \underset{0}{\cancel{c_2(\mathcal{E})}} = 0$$

$f^2 = 0$

$\xi . f = 1 = \deg \mathcal{O}_X(1)|_F$

$$N^1(X) = \mathrm{Span}(f, \xi)$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Want to compute $\overline{NE}(X)$

$E \subseteq X$ effective divisor

nonzero

$\mathcal{Y} \in \mathrm{Pic}(C)$

$$E \in |H^0(X, \mathcal{O}_X(m) \otimes \pi^* \mathcal{L})|$$

$$m\xi + \underline{(\deg \mathcal{L})} f = [E]$$

<u>Projection formula:</u>

$$H^0(\mathcal{O}_X(m) \otimes \pi^*\mathcal{L}) = H^0(C, \pi_* \underbrace{(\mathcal{O}_X(m) \otimes \pi^*\mathcal{L})}_{= \pi_* \mathcal{O}_X(m) \otimes \mathcal{L}})$$

$$\pi_* \mathcal{O}_X(m) = \mathrm{Sym}^m(\mathcal{E})$$

by the proj definition of $\mathbb{P}_C(\mathcal{E})$

$$H^0(\mathcal{O}_X(m) \otimes \pi^*\mathcal{L}) = H^0(C, \mathrm{Sym}^m(\mathcal{E}) \otimes \mathcal{L})$$

<u>Fact</u> if $g(C) \geq 2$, then there exist <u>stable</u> bundles $\mathcal{E}$

$$\Longrightarrow H^0(C, \mathrm{Sym}^m(\mathcal{E}) \otimes \mathcal{L}) \neq 0$$

$$\Longrightarrow \deg \mathcal{L} > 0 \qquad m > 0$$

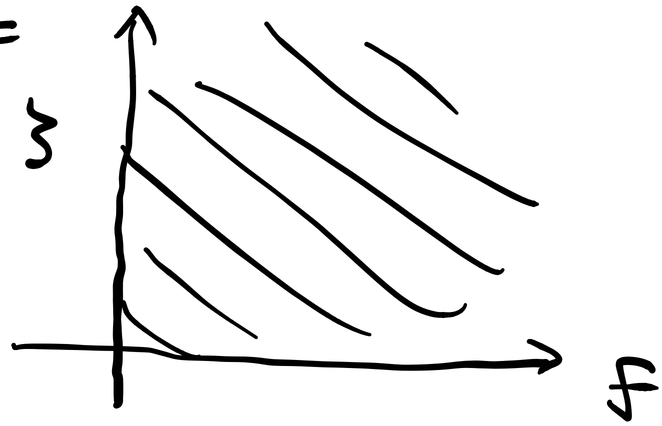
Narasimhan - Seshadri

$\Longrightarrow$ on X, $\quad [E] = m\xi + df$

$d > 0$

$m \geq 0$

$\Longrightarrow \quad \overline{NE}(X) =$



<u>Furthermore</u>

$$\xi . E = d > 0$$

$$\Longrightarrow \quad \xi > 0 \text{ on } NE(X) \setminus \{0\}$$

but $\xi^2 = 0$ so ξ <u>not ample</u>

<u>Upshot</u> need $L^2 > 0$ Nakai-Moishezon

need $\overline{NE}(X)$ in Kleiman

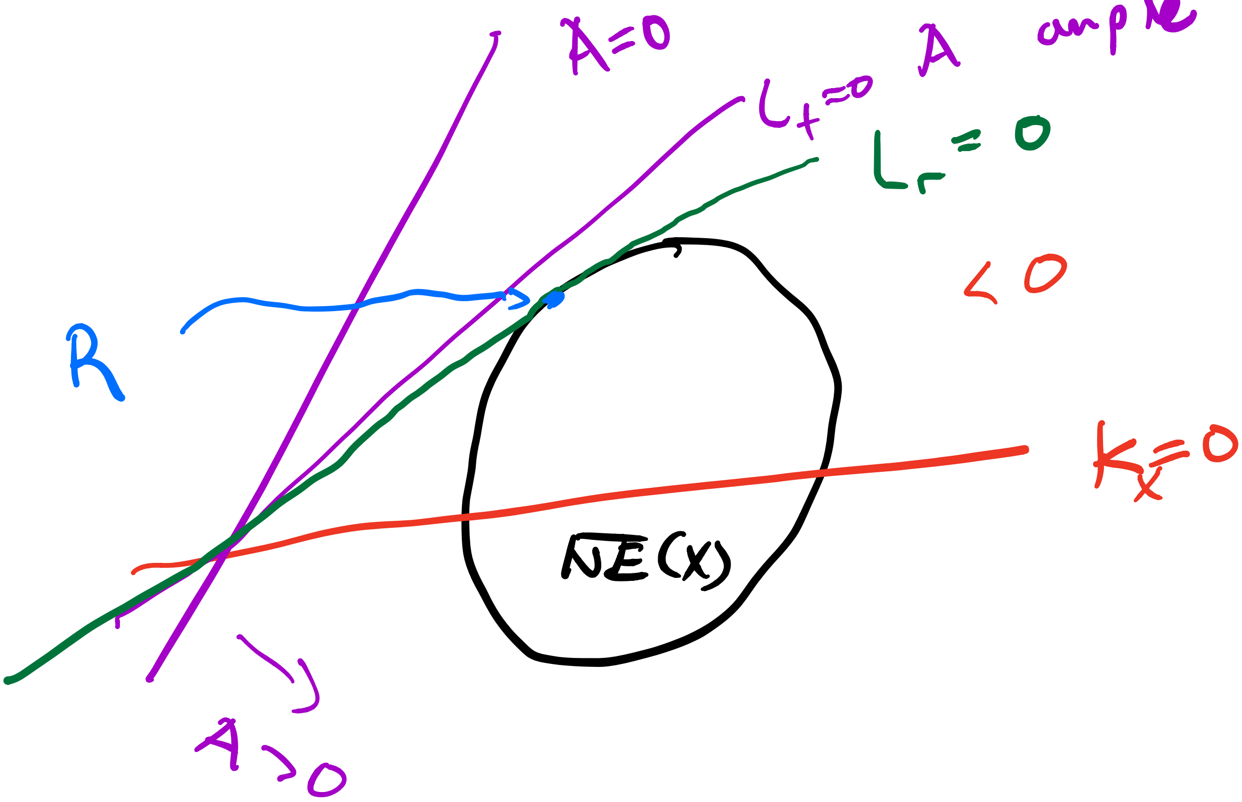
Furthermore, $NE(X) \neq \overline{NE}(X)$

$\xi \in \overline{NE}(X)$ $\qquad \deg \mathcal{L} = 0$

but $H^0(X, \mathcal{O}_X(m)) = H^0(C, \mathrm{Sym}^m(\mathcal{E}) \otimes \mathcal{L}) = 0$

So γ is <u>not</u> effective

§2: Surfaces w/ K_X <u>not</u> nef



$L_t = tK_X + A$
$t>0$

r = nef threshold
$= \sup_t \{ tK_X + A \text{ is nef} \}$

Thm (Rationality)

r is a rational number

so mL_r is the class of a line bundle for some m divisible enough

Thm (Base-Point Free)

if A is ample & $A + rK_X = L$ is nef for r rational, then L is semiample

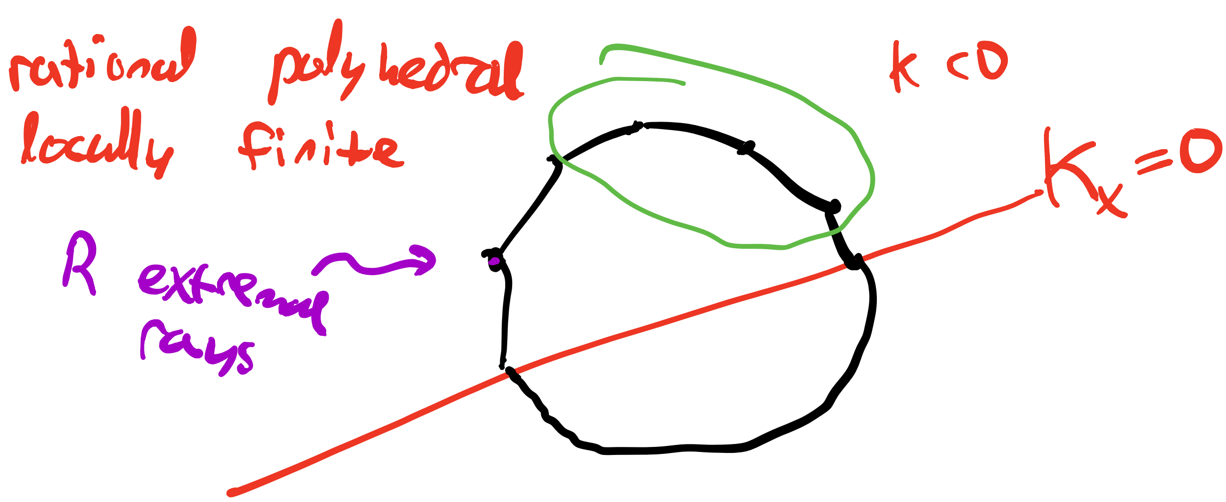
$\varphi_{|dL|}$ is a morphism for d large + divisible

$\varphi_{|dL|} : X \to Z$ contracts exactly the curves C s.t. $L \cdot C = 0$

$\iff [C] \in R$

If we pick A generic enough then $\{L = 0\} \cap \overline{NE}(X)$ will be a ray

Cone + Contraction theorem



R extremal if $n+w \in R \Rightarrow n, w \in R$ whenever $n+w \in R$, $\wedge$ cone

+ for each R, there exists a morphism $\varphi_R : X \to Z$

s.t. $\varphi_R(C) = pt \iff [C] \in R$

Back to surfaces, we won't need cone + contraction, just rationality + bpf

Suppose K_X not nef, pick A generic ample

$L = rK_X + A$ $\quad r =$ nef threshold

$\{L=0\} \cap \overline{NE}(X) = R$ extremal

$\varphi = \varphi_{|dL|} : X \to Z$

Classify possibilities for φ in terms of dim Z