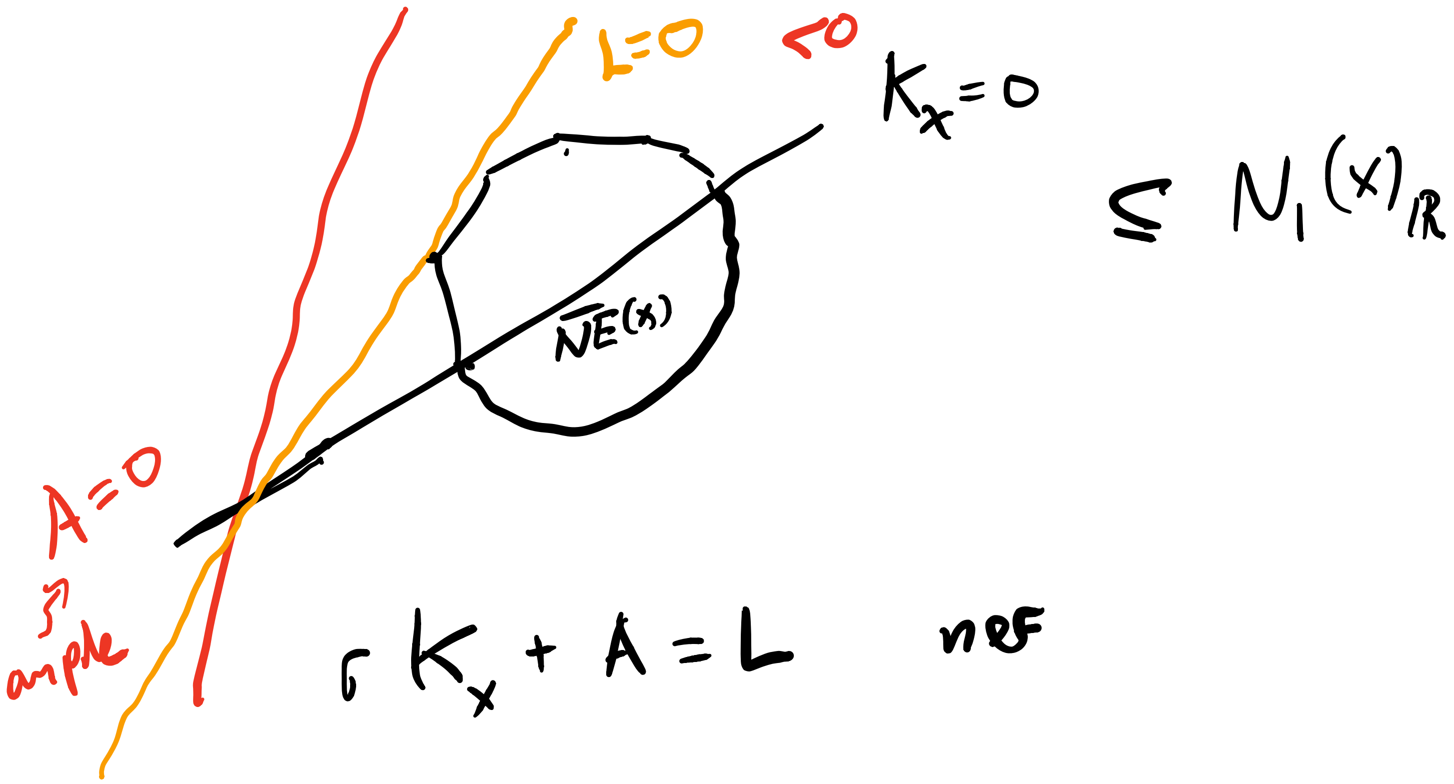
§1: Surfaces w/ K_X is not nef



$$rK_X + A = L \quad \text{nef}$$

Thm (rationality)

$$r := \sup_t \{ tK_X + A \text{ is nef} \} \in \mathbb{Q}$$

Thm (base point free theorem)

if $L = rK_X + A \quad r \in \mathbb{Q}$

& L is nef $\Longrightarrow$ L is semiample

Pick A s.t. $\{L=0\} \cap \overline{NE}(X)$

$= R = \mathbb{R}_{\geq 0} v$

if $n_1 + n_2 \in R \Longrightarrow n_i \in R$

$\cap$
$\overline{NE}(X)$

by bpF

$\psi := \varphi_{|lL|} : X \longrightarrow Z$

C contracted by $\varphi \Longleftrightarrow [C] \in R$

$\Longrightarrow$ C have to be irreducible

C_1, C_2 can't be numerically eq to rC

dim Z	
2	$C \subseteq \varphi^{-1}(p)$ $p \in Z$ $\Longrightarrow$ $C^2 < 0$ (Hodge index) $K_X . C < 0$ $\xrightarrow[\text{adjunction}]{}$ $C \cong \mathbb{P}^1$ $C^2 = -1$

	running the argument from Castelnuovo $\Rightarrow$ φ is a blowdown
1	$\varphi : X \to Z$ ← smooth fibers of φ are irreducible $F^2 = 0$ + $K_X \cdot F < 0$ $\Rightarrow$ $F_{red} \cong \mathbb{P}^1 \cong F$ i.e. fibers are reduced (Exc) φ is Fano fibration its a projective bundle $\mathbb{P}_C(\mathcal{E})$
0	$\varphi : X \to Z = pt$ $m(A + rK_X) \sim \varphi^* \mathcal{O}_{pt} = 0$ $-mrK_X \sim mA$ is ample $\Rightarrow$ X is Fano since $-K_X$ is ample

by extremality of R

$\Rightarrow \rho(X) = 1$

$\Rightarrow X = \mathbb{P}^2$

Thm: X smooth projective surface

s.t. $X = X_0 \to X_1 \to \dots \to X_m$ minimal

K_{X_m} not nef.

Then either 1) $X_m \to C$ is ruled

2) $X_m = \mathbb{P}^2$

Moreover,

$\kappa(X) = \kappa(X_m)$
$= -\infty$

Pf) $X_0 \to \dots \to X_m$

Since K_{X_m} not nef, $\exists\ \varphi : X_m \to Z$

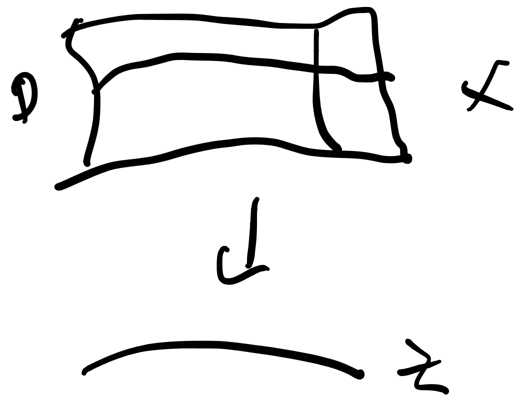
$\varphi : X_m \to Z$ either a $\mathbb{P}^1$ bundle or X_m Fano

Case 1

Z = smooth curve, $k = \bar{k}$

Tsen's theorem $\Rightarrow$
$$\begin{tikzcd} (X_m)_\eta \arrow[r] & k(Z) \arrow[l, bend right] \end{tikzcd}$$

$\Rightarrow (X_m)_\eta = \mathbb{P}^1_{k(Z)}$



$$\Rightarrow \begin{tikzcd} X_m \arrow[r, "\varphi"'] & Z \arrow[l, bend right, "s"'] \end{tikzcd}$$

$D = s(Z)$, $\quad D . F = 1$

i.e. D is horizontal

Want for each point $p \in Z$

$p \in U \subseteq Z$ open

$$\begin{tikzcd} X \arrow[d, "\varphi"'] & \supseteq & \varphi^{-1}(U) \arrow[r] \arrow[d] & \mathbb{P}^1 \times U \arrow[d] \\ Z & \supseteq & U \arrow[r, equal] & U \ni p \end{tikzcd}$$

$F = \varphi^{-1}(p)$

$$0 \to \mathcal{O}_X(D + (r-1)F) \to \mathcal{O}_X(D + rF) \to \underset{\substack{\cong \\ \mathcal{O}_{\mathbb{P}}(1)}}{\mathcal{O}_F(D)} \to 0$$

__Claim__ For $r \gg 0$

$$V \subseteq H^0(\mathcal{O}_X(D+rF)) \twoheadrightarrow H^0(\mathcal{O}_F(D)) = H^0(\mathcal{O}_{\mathbb{P}^1}(1))$$

__Exc__

Pick V 2-dimensional mapping isomorphically to $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$

$$\varphi_{|V|} : X \dashrightarrow \mathbb{P}^1$$

$$V|_F = H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \hookleftarrow \text{ bpf on } F$$

$\Rightarrow \exists\, \varphi^{-1}(U) \supseteq F$ up to shrinking

s.t. $\varphi_{|V|}\big|_{\varphi^{-1}(U)}$ is a morphism

$$\begin{tikzcd}
\varphi^{-1}(U) \arrow[r, "\varphi_{|V|}"] & \mathbb{P}^1 \\
F \arrow[u, phantom, sloped, "\subseteq"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\varphi_{|V|} \times \varphi : \varphi^{-1}(U) \arrow[r] \arrow[dr] & \mathbb{P}^1 \times U \arrow[d] \\
& U
\end{tikzcd}$$

$\Rightarrow \varphi_{|V|} \times \varphi$ is an isomorphism on a nbhd of F

Case 2

$\varphi : X \to pt$ $-K_X$ ample

φ contracted an extremal ray

1) $\Rightarrow \rho(X) = \dim N^1(X)_{\mathbb{R}} = 1$

2) $H^i(X, \mathcal{O}_X) = H^i(X, \mathcal{O}_X(K_X + (-K_X)))$ Ample

$i > 0$ $= 0$ by KV

Kodaira vanishing theorem

$H^i(X, K_X + A) = 0 \quad i > 0$

for A ample

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp(2\pi i \cdot)} \mathcal{O}_X^* \to 0$$

$$\underset{\substack{\| \\ 0}}{H^1(X, \mathcal{O}_X)} \to \underset{\substack{\| \\ \mathrm{Pic}(X)}}{H^1(\mathcal{O}_X^*)} \xrightarrow{\sim} H^2(X, \mathbb{Z}) \to \underset{\substack{\| \\ 0}}{H^2(X, \mathcal{O}_X)}$$

So $\mathrm{Pic}(X) \xrightarrow{\sim} H^2(X, \mathbb{Z})$

$$\underset{\substack{\| \\ \mathbb{Z} \oplus \text{tors}}}{\mathrm{Pic}(X)} \to N^1(X) \otimes \mathbb{Q} \xrightarrow{\sim} H^2(X, \mathbb{Q})$$

So numerical equivalence $\Longleftrightarrow$ $\mathrm{Pic}(X)/\text{tors}$

$$N^1(X)_{\mathbb{Z}} = \mathrm{Pic}(X)/\text{tors} = \mathbb{Z}$$

$\Rightarrow \exists$ H <u>ample</u> which generates $\mathrm{Pic}(X)/\text{tors} \cong H^2(X, \mathbb{Z})/\text{tors}$

$-K_X = rH$ for some r

<u>Fact</u> since H generates $H^2(X, \mathbb{Z})/\text{tors}$

$\Rightarrow H^2 = 1$

<u>$r > 1$</u>

$$h^0(\mathcal{O}_X(H)) \underset{=}{\geq} \frac{1+r}{2} + 1 \underset{\simeq}{\geq} \underset{3}{2}$$

$$h^1(\mathcal{O}_X(H)) = 0$$

$$h^2(\mathcal{O}_X(H)) = h^0(\mathcal{O}_X(K_X - H))$$

$$= h^0(\mathcal{O}_X(dK_X)) = 0$$

since $-K_X$ is ample

$D \in |H|$ a section

by adjunction, $h^1(D, \mathcal{O}_D) = \frac{1}{2}(1-r)+1 < 1$
$= 0$

So $D \cong \mathbb{P}^1 \Rightarrow r = 3$

$$H^0(X, \mathcal{O}_X(D)) \to H^0(D, \mathcal{O}_X(D)|_D) \to \underline{H^1(X, \mathcal{O}_X)}$$

$$H^0(D, \mathcal{O}_X(D)|_D) = \mathcal{O}_{\mathbb{P}^1}(1), \qquad H^1(X, \mathcal{O}_X) = 0$$

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_X(D)|_D \to 0$$

$$H^1(\cancel{X}, \mathcal{O}_X(D)) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 0$$

$\Longrightarrow |H|$ is bpf

so we have

$$\varphi_{|H|} : X \to \mathbb{P}(|H|) = \mathbb{P}^2 \qquad \underline{h^0 = 3}$$

$$\varphi^*\mathcal{O}_{\mathbb{P}^2}(1) = H$$

$$(\varphi^*\mathcal{O}_{\mathbb{P}^2}(1))^2 = H^2 = 1$$

so φ is degree one

$\Longrightarrow$ φ is an isomorphism by ZMT

<u>$r=1$</u> $\quad -K_X = H \qquad H^2 = 1$

Use Noether's Formula (computation of RR using topological inv)

$$e_{top}(X) = 3$$

$$\chi(\mathcal{O}_X) = \frac{1}{12}\Big(\underbrace{(-K_X)^2}_{=1} + \underbrace{e(X)}_{=3}\Big) = \frac{1}{3}$$

gives contradiction

Finally, use that $\kappa(\mathbb{P}^2) = -\infty$

and that $\kappa(Z \times \mathbb{P}^1) = -\infty$

+ birational invariance of κ.

Cor (Castelnuovo's Rationality Criteria)

X is rational $\iff h^1(X, \mathcal{O}_X) = h^{0,1}$
$= h^0(X, \mathcal{O}_X(2K_X)) = P_2$
$= 0$

birational to $\mathbb{P}^2$

$$\boxed{P_n = H^0(X, \mathcal{O}_X(nK_X)) \quad \text{Plurigenus}}$$

Pf sketch

$\Longrightarrow$ $h^{0,1}$ & P_2 are birational invariants!

exercise

$h^{0,1}(X) = h^{0,1}(Y)$ for $X \dashrightarrow Y$ smooth proj birational surfaces

$P_2(X) = P_2(Y)$

$Y = \mathbb{P}^2$ the #'s are zero

$\Longleftarrow$

Step 1 K_X is not nef

$$\chi(\mathcal{O}_X) = \underset{\substack{\| \\ 1}}{h^0(\mathcal{O}_X)} - \underset{\substack{\| \\ 0}}{h^1(\mathcal{O}_X)} + \underset{\substack{\| \\ h^0(\mathcal{O}_X(K_X)) \\ \| \\ 0}}{h^2(\mathcal{O}_X)}$$

$$= 1$$

$$h^2(\mathcal{O}_X(-K_X)) = h^0(\mathcal{O}_X(2K_X)) = 0$$

$$(D)(D-K)$$

$$h^0(\mathcal{O}_X(-K_X)) \geq \frac{1}{2}(-K_X)(-K_X-K_X) + \chi(\cancel{\mathcal{O}_X}) \; 1$$

$$= K_X^2 + 1$$

__Lemma__ if D is nef

$\Rightarrow$ $D^2 \geq 0$

$D.A \geq 0$ $\quad A$ ample

__Pf__ $D.A \geq 0$ since A effective

$\varepsilon \in \mathbb{R}$

__Kleiman__ $(D + \varepsilon A)$ ample

$$(D+\varepsilon A)^2 > 0$$

$$D^2 = \lim_{\varepsilon \to 0} (D+\varepsilon A)^2 \geq 0$$

Suppose K_X is nef so $K_X^2 \geq 0$

then $h^0(\mathcal{O}_X(-K_X)) \geq 1$

so $D \in |-K_X|$

$A.K_X \geq 0$

$$0 < A.(-K_X) = -A.K_X \leq 0$$

Contradiction

Run MMP

$$X = X_0 \to X_1 \to \cdots \to X_m \xrightarrow{\varphi} Z$$

contract (-1) curves

X_m still satisfies

$h^{0,1} = P_2 = 0 \Rightarrow K_{X_m}$ not nef

$\Rightarrow \quad X_m = \mathbb{P}^2$ ✓

or $\quad \varphi : X_m \to Z \quad$ ruled

$X_m \overset{\sim}{\dashrightarrow} Z \times \mathbb{P}^1$

$$0 = h^1(X_m, \mathcal{O}_{X_m}) = h^1(Z \times \mathbb{P}^1, \mathcal{O}_{Z \times \mathbb{P}^1})$$

$$= h^1(Z, \mathcal{O}_Z) = 0$$

$\Rightarrow \quad X_m \xrightarrow{\sim} \mathbb{P}^1 \times \mathbb{P}^1$