Surfaces with K_X nef

§1: Def a minimal model is a smooth projective surface X with K_X nef.

$\Rightarrow$ minimal surfaces

Thm (Nonvanishing)
Suppose K_X is nef, then for some $m>0$,

$$H^0(X, \mathcal{O}_X(mK_X)) \neq 0$$

Cor if K_X nef, then $\kappa(X) \geq 0$

Cor (Coarse classification)
every surface X is birational to a minimal surface X_m where

- X_m is a minimal model

$\iff \kappa(X) \geq 0$

- X_m is ruled or $\mathbb{P}^2$

$\iff \kappa(X) = -\infty$

Thm (Abundance)

If K_X is nef, then K_X semiample

$|mK_X|$ bpf for some m

Rmk Nonvanishing + abundance

$\implies$ classification of surfaces by Kodaira dimension

Essentially need to classify to prove these theorems

§2: Background

Big divisor, Iitaka dimension (Positivity in AG I by Lazarsfeld)

L is a Cartier divisor on X

$$\kappa(X, L) := \max_{m \in N(L)} \{\dim \phi_{|mL|}(X)\} \leq \dim(X)$$

$N(L) := \{ m \mid |mL| \neq \emptyset \}$ numerical semigroup of L

$\kappa(X, K_X) = \kappa(X)$

Prop L has Iitaka dim κ

$\iff$ there exist a, A s.t. for all $m \in N(L)$ large

$$a m^{\kappa} \leq h^0(X, \mathcal{O}_X(mL)) \leq A m^{\kappa}$$

Def L is big if $\kappa(X, L) = \dim X$

$\iff$ $h^0(X, \mathcal{O}_X(mL)) \geq C m^{\dim X}$ for $m \in N(L)$

Kodaira's Lemma

Suppose D is big & E effective, then $H^0(X, \mathcal{O}_X(mD - E)) \neq 0$ for $m \in N(L)$ large

Cor TFAE

1) D is big
2) $\exists$ an ample A s.t.
$$D = A + N \qquad N \text{ effective}$$

Cor Suppose D is nef then
$$D \text{ is big} \iff D^{\dim X} > 0$$

Cor X is a minimal model in dim 2
K_X is nef
then $\kappa(X) = 2 \iff K_X^2 > 0$

Albanese morphism

Thm let X smooth projective
then $\exists$ abelian variety $\mathrm{Alb}(X)$
& a morphism
$$\alpha : X \to \mathrm{Alb}(X)$$
1) if $\beta : X \to T$ with T abelian,

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[dr, "\alpha"'] & T \\ & \mathrm{Alb}(X) \arrow[u, dashed] \end{tikzcd}$$

$\exists$

2) $\alpha^*: H^0(\mathrm{Alb}(X), \Omega^1_{\mathrm{Alb}(X)}) \xrightarrow{\sim} H^0(X, \Omega^1_X)$

is an isomorphism

$\dim \mathrm{Alb}(X) = \dim H^0(X, \Omega^1_X)$
$= h^{1,0} = q(X) \leftarrow$ irregularity
$= h^{0,1}$

3) $\mathrm{Alb}(X)$ is generated by $\alpha(X)$

4) if

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha_X"'] & Y \arrow[d, "\alpha_Y"] \\ \mathrm{Alb}(X) \arrow[r, "\alpha(f)"] & \mathrm{Alb}(Y) \end{tikzcd}$$

f is surjective $\Rightarrow \alpha(f)$ surjective

5) $\alpha: X \to C = \alpha(X)^{norm} \to \mathrm{Alb}(X)$

if α factors through a smooth curve C, then $\mathrm{Alb}(X) = \mathrm{Jac}(C)$

$g(C) = h^0(X, \Omega^1_X) = q(X)$

Sketch of construction

$$T = V/\Gamma$$

V $\mathbb{C}$-vector space
Γ integer lattice

$\Big\}$ complex torus

polarization — to make it projective

Fact any $u : T_1 \to T_2$ is up to translation is induced by $a : V_1 \to V_2$ linear map s.t. $a(\Gamma_1) \subseteq \Gamma_2$

$$\Longrightarrow \quad u^* : H^0(T_2, \Omega^1_{T_2}) \to H^0(T_1, \Omega^1_{T_1})$$

$$\begin{tikzcd} H^0(T_2, \Omega^1_{T_2}) \arrow[r, "u^*"] \arrow[d, equal] & H^0(T_1, \Omega^1_{T_1}) \arrow[d, equal] \\ V_2^* \arrow[r, "a^*"] & V_*^1 \end{tikzcd}$$

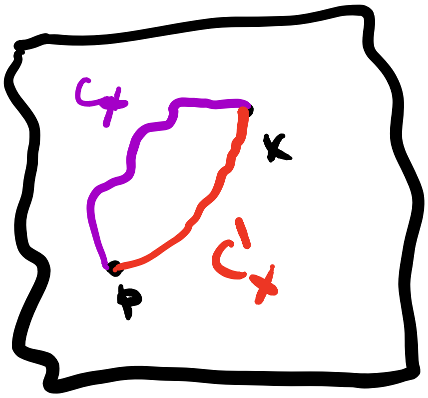
$$\mathrm{Alb}(X) = H^0(X, \Omega^1_X)^* \big/ H$$

Polarization comes from Hodge theory

$$H = \mathrm{im}(H_1(X, \mathbb{Z}) \to H^0(X, \Omega^1_X)^*)$$

$$\gamma \longmapsto \left(\omega \mapsto \int_\gamma \omega \in \mathbb{C}\right)$$

$$\alpha_x : X \longrightarrow \mathrm{Alb}(X) \qquad \alpha(x) = \left(\omega \mapsto \int_{c_x} \omega\right)$$



$$c_x^{-1} \circ c'_x \in H_1(X, \mathbb{Z})$$

§3: Nonvanishing

X is a minimal model in dim 2

then $H^0(X, \mathcal{O}_X(mK_X)) \neq 0$

Lemma K_X is nef & $\kappa(X) \leq 0$

then $\chi(\mathcal{O}_X) \geq 0$

Pf K_X nef but not big so

$K_X^2 = 0$

Noether's formula

$$12\chi(\mathcal{O}_X) = \overset{0}{\cancel{K_X^2}} + e(X)$$ Betti #'s

$$= e(X) = b_0 - b_1 + b_2 - b_3 + b_4$$

$e(X) = e_{top}(X)$ topological Euler #

$$= 2(1-b_1) + b_2 \qquad (**)$$

$$\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) \qquad (*)$$

$$h^0(\mathcal{O}_X) = 1, \qquad h^2(\mathcal{O}_X) = h^0(\mathcal{O}_X(K_X)) \leq 1$$

$$b_1 = h^1(X, \mathcal{O}_X) + h^0(X, \Omega^1_X)$$
$$= 2h^1(X, \mathcal{O}_X)$$ Hodge theory

$$\chi(\mathcal{O}_X) \leq 2 - h^1(\mathcal{O}_X) \quad \text{by } (*)$$

$$12\chi(\mathcal{O}_X) = 2 - 2h^1(X, \mathcal{O}_X) + b_2$$
$$= 2 - 4h^1(X, \mathcal{O}_X) + b_2$$
$$\geq 2 + 4\chi(\mathcal{O}_X) - 8 + b_2$$
$$= -6 + 4\chi(\mathcal{O}_X) + b_2$$

$$8\chi(\mathcal{O}_X) \geq -6 + b_2 \geq -6$$

$$\chi(\mathcal{O}_X) \geq -6/8$$

$$\geq 0$$

Proof of nonvanishing

Suppose K_X is nef

but $\kappa(X) = -\infty$

then by Lemma above, $\chi(\mathcal{O}_X) \geq 0$

$$= 1 - h^1(X, \mathcal{O}_X) + \underbrace{h^0(\cancel{X, \mathcal{O}_X(K)})}_{\substack{= \\ P_1}} \quad 0$$

$$\Longrightarrow \quad 1 - h^1(X, \mathcal{O}_X) \geq 0$$

Also know $P_m = h^0(X, \mathcal{O}_X(mK_X)) = 0$

in particular, $P_2 = 0$

Castelnuovo's criteria: $P_2 = h^1(X, \mathcal{O}_X) = 0$

$$\Longrightarrow \quad X \text{ rational}$$

can't happen b/c K_X is nef

$$\Rightarrow h^1(X, \mathcal{O}_X) > 0 \Rightarrow h^1(X, \mathcal{O}_X) = 1$$

$$h^1(X,\mathcal{O}_X) = h^0(X, \Omega^1_X)$$

$\alpha : X \to \mathrm{Alb}(X) = E$ ← elliptic curve

α surjective + connected fibers $\Rightarrow$ flat

$F \subseteq X$ a general fiber is a smooth curve $g(F) = g$

<u>Claim 1</u> $g(F) \geq 1$

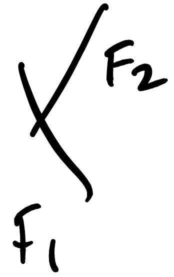
$F^2 = 0$

since K_X is nef

$$K_F = (K_X + F)|_F \geq 0$$

but if $\deg K_F \geq 0 \Rightarrow g(F) \geq 1$

<u>Claim 2</u> α is smooth i.e. fibers are smooth



step 1 suppose $\alpha^{-1}(p) = F_1 + F_2$

let H be an ample,

H, F_1, F_2 are linearly independent numerical classes

$$\Rightarrow \quad 3 \le \rho(X) \le \dim H^2(X, \mathbb{R}) = b_2$$

$$0 = e(X) = 2 - 4h^1(\cancel{X,}\mathcal{O}_X) + b_2$$

$b_2 = 2$ contradiction

$\Rightarrow \quad \alpha^{-1}(p)$ is irreducible

Step 2 $\alpha^{-1}(p)_{red} = F_p \hookleftarrow$ irreducible

$$\alpha^{-1}(p) = m_p F_p \equiv F$$

Lemma $e(F_p) \geq 2\chi(\mathcal{O}_{F_p})$

with equality $\Longleftrightarrow$ F_p smooth

$2 - 2g = 2\chi(\mathcal{O}_F)$ by RR/SD in smooth case

Use $n: X^n \to X$ normalization

Compute w/ adjunction:

$$e(F_p) \geq 2\chi(\mathcal{O}_{F_p}) = 2\frac{1}{m_p}\chi(\mathcal{O}_F)$$

$$= \frac{1}{m_p} e(F) \geq e(F)$$

$g(F) \geq 1$

$\Rightarrow \quad e(F) \leq 0$

discriminant of α

$$0 = e(X) = e(F)e(E \setminus \Delta) + \sum_{p \in \Delta} e(F_p)$$

$$= e(F)\cancel{e(E)}^{\,0} + \sum_{p \in \Delta} \underbrace{(e(F_p) - e(F))}_{\geq 0}$$

E elliptic curve

$$\Longrightarrow \quad e(F_p) = e(F) \quad \text{for all } p$$
$$= \frac{1}{m_p} e(F)$$

So either

1) $e(F) = 0$ and m_p arbitrary
$g(F) = 1$

2) $g(F) \geq 2$ & $m_p = 1$

$\Rightarrow e(F_p) = 2\chi(\mathcal{O}_{F_p})$

so F_p smooth

<u>Thm</u> (Kodaira's Canonical bundle Formula)

let $f: X \to C$ be a minimal genus 1 fibration

no (-1) curves in the fiber

$g(F) = 1$

$m_1 F_{p_1}, \dots, m_n F_{p_n}$ are nonreduced fibers w/ $m_i > 1$

$$\omega_X = f^*\big(\underbrace{\omega_C \otimes (R^1 f_* \mathcal{O}_X)^\vee}_{\deg = 0}\big) \otimes \mathcal{O}_X\big(\textstyle\sum (m_i - 1) F_{p_i}\big)$$

deg=0 in our case

if $m_i \to$

$$\deg (R^1 f_* \mathcal{O}_X)^\vee = \chi(\mathcal{O}_X)$$

For us, $\chi(\mathcal{O}_X) = 0$

$NF_p = mF$

$C = E$ so $\omega_E = \mathcal{O}_X$

$$\Rightarrow \omega_X^{\otimes N} = \alpha^*(\deg 0 + \underbrace{\deg > 0}_{\text{ample on a curve}})$$

$\Rightarrow$ has a lot of sections

Contradiction $\Longrightarrow$ $m_p = 1$ even in $g=1$ case so α is smooth.

Step 3

$$\begin{array}{c} X \\ \alpha \downarrow \\ E \end{array}$$

α is a smooth family of genus ≥ 1 curves, $g(E) = 1$

Thm: If $\alpha: X \to E$ is a smooth + proper morphism w/ $g(E) \leq 1$ & general fibers F

$g(F) \geq 2$ α is isotrivial:

$$\begin{tikzcd} F \times E' \cong X' \arrow[r] \arrow[d, "\alpha'"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\alpha"] \\ E' \arrow[r, "t"] & E \end{tikzcd}$$

$\leftarrow$ finite étale

$g=1$ $\alpha_* \omega_{X/E}$ is a torsion line bundle

$\leftarrow$ finite étale

$t: E' \to E$ s.t.

$$\mathcal{O}_{E'} = t^* \alpha_* \omega_{X/E} = \alpha'_* \omega_{X'/E'}$$