Minimal Models in dim 2

Def X s.t. K_X is nef

Thm (Nonvanishing)
If X is a minimal model, then $H^0(X, mK_X) \neq 0$ for some $m > 0$

Pf Suppose we had a minimal model X but w/ $\kappa(X) = -\infty$

$\Rightarrow \alpha : X \to E$ ← elliptic curve

α is smooth & $g(F) \geq 1$

Thm $\alpha : X \to E$ smooth proper w/ fibers $g(F) \geq 1$, $g(E) \leq 1$ then α is isotrivial:

$g \geq 2$

$$\begin{tikzcd} E' \times F \cong X' \arrow[r, "t'"] \arrow[d, "\alpha'"'] & X \arrow[d, "\alpha"] \\ E' \arrow[r, "t"] & E \end{tikzcd}$$

where t is étale

$\underline{g=1}$ $\alpha_* \omega_{X/E}$ this is a torsion line bundle

$\rightsquigarrow t: E' \to E$ $\mathcal{O}_{E'} \cong t^*\alpha_*\omega_{X/E} \cong \alpha'_*\omega_{X'/E'}$

t étale

PF sketch

$\underline{g \geq 2}$

$$\begin{tikzcd} \mathbb{C}, \mathbb{P}^1 = \tilde{E} \arrow[r, "\exists \text{ lift}", "\text{holomorphic}"'] \arrow[d] & \mathcal{H}_g \arrow[d, "\text{complex analytic quotient}"] \\ E \arrow[r] & \mathcal{M}_g \cong \mathcal{H}_g/\Gamma_g \end{tikzcd}$$

holomorphic

bounded

complex analytic quotient

Since $\mathcal{H}_g$ is bounded, this lift is constant

$\Rightarrow \tilde{X} = \tilde{E} \times_E X \xrightarrow{\tilde{\alpha}} \tilde{E}$ is a product

$\tilde{E} \times_E X = F \times \tilde{E}$

but $\mathrm{Aut}(F)$ finite $\Rightarrow$ factor through a finite cover $E' \to E$

$\underline{g=1}$ $E \xrightarrow{\mathcal{H}} \text{"}\mathcal{M}_{g,1}\text{"}$ = Space of genus 1 curve w/ a point

$X \xrightarrow{\alpha} E$ might not have a section

$\wr$

$$\begin{tikzcd} \mathrm{Jac}(X) \arrow[r, "J(\alpha)"] & E \arrow[l, bend left] \end{tikzcd}$$

<u>Rmk</u> Start of a beautiful story about hyperbolicity of moduli, positivity of $\alpha_* \omega_{X/E}$

$\rightsquigarrow$ Iitaka's conjecture on subadditivity of Kodaira dimensions

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<u>Thm</u> (Iitaka)

Suppose $h: X' \to X$ finite unramified

$\Rightarrow$ $\kappa(X') = \kappa(X)$

<u>$g \geq 2$</u> $\kappa(X) = \kappa(X') = \kappa(F \times E') \geq 1$

$g(F) \geq 2$ $\boxed{\text{Contradiction!}}$

$g(E') \geq 1$

<u>$g=1$</u>

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\alpha'"'] & X \arrow[d, "\alpha"] \\ E' \arrow[r, "t"] & E \end{tikzcd}$$

$$t^* \alpha_* \omega_{X/E} = \alpha'_* \omega_{X'/E} = \mathcal{O}_{E'}$$

$$\alpha'_* \omega_{X'} = \alpha'_* \left((\alpha')^* \omega_{E'} \otimes \omega_{X'/E'} \right)$$

$$\cong \omega_{E'} \otimes \underbrace{\alpha'_* \omega_{X'/E'}}_{\mathcal{O}_{E'}} = \omega_{E'}$$

Projection formula

$g(E') \geq 1$

$$\Rightarrow \quad H^0(X', \mathcal{O}_{X'}(K_{X'})) = H^0(E', \omega_{E'}) = g(E) \geq 1$$

$$\Rightarrow \quad \kappa(X) = \kappa(X') \geq 0$$

$\boxed{\text{Contradiction!}}$

§2: Abundance in dim 2

<u>Thm</u> If X is a minimal model in dim 2, then K_X is semiample

mK_X is bpf

K_X nef $\Rightarrow$ K_X semiample

Proof by nonvanishing $\kappa(X) \geq 0$

$\kappa = 2$ (general type)

It's a corollary of the full base point free theorem

Thm X smooth proj variety

$D = \sum d_i D_i \qquad 0 < d_i \leq 1$ rational

+ singularities of D are "nice" (X, D) has klt singularities

If L nef Cartier divisor such that $aL - (K_X + D)$ is ample for some $a > 0$

$\Longrightarrow$ mL bpf for some $m \gg 0$

$\kappa = 1$

$K_X^2 = 0, \qquad K_X \not\equiv 0$ (numerical Kodaira dim)

nef but not big

$|mK_X| = |M| + F$ ← fixed part

$\neq \emptyset$

moving part of mK_X

M nef

$$0 \leq M^2 \leq M.(M+F) \leq (M+F)^2 = (mK_X)^2 = 0$$

$$\Longrightarrow \quad M^2 = M.F = F^2 = 0$$

moving divisor M w/ $M^2 = 0$

$\Longrightarrow$ $|M|$ is base point free

b/c $\bigcap_{M' \in |M|} M' = \emptyset$

$\phi_{|M|} : X \longrightarrow Z$ ← curve

F satisfying

1) $F.M = 0$

$\Longrightarrow$ $F \leq$ fiber of ϕ

2) $F^2 = 0$ ← Hodge index

$\Longrightarrow$ $F = \sum m_i F_i$

F_i are fibers of ϕ

$$mK_X = M + F = \phi_{|M|}^* H_Z + \sum m_i \phi^* p_i$$

$p_i \in Z$

$$= \phi^*(\underbrace{H_Z + \sum m_i p_i}_{\text{ample divisor}})$$

pull back of ample by a morphism is bpf

$K = 0$

$h^0(X, \mathcal{O}(mK_X)) \leq 1$ for any $m > 0$

$\chi(\mathcal{O}_X) \geq 0$ $p_1 = p_g$ ← geometric genus

$$\overset{\shortparallel}{1 - h^1(\mathcal{O}_X) + h^0(\mathcal{O}_X(K_X))} \geq 0$$

irregularity → $h^{0,1} = h^{1,0} = q(X)$ ≤ 1

Cases

i) $p_g = q = 0$ $2K \sim 0$ Enriques

ii) $p_g = 0,\ q = 1$ $mK_X \sim 0$ $m > 0$ bielliptic

iii) $p_g = 1,\ q = 0$ $K_X \sim 0$ K3 surface

iv) $p_g = q = 1$ doesn't exist

v) $p_g = 1$ $q = 2$ abelian surface

Need to check that

$mK_X \sim 0$ for some $m > 0$

i) $P_g = q = 0$, $\chi(\mathcal{O}_X) = 1$

<u>Castelnuovo's Rationality</u>

if $P_2 = H^0(X, \mathcal{O}_X(2K_X)) = 0$

$\Rightarrow X$ rational, can't happen

$\Longrightarrow H^0(X, \mathcal{O}_X(2K_X)) \neq 0$

want $h^0(X, \mathcal{O}_X(-2K_X)) \geq 1$

$\Longrightarrow 2K_X \sim 0$

$$h^0(X, \mathcal{O}_X(-2K_X)) + \underbrace{\cancel{h^2(X, \mathcal{O}_X(-2K_X))}}_{= h^0(\mathcal{O}_X(3K_X)) \leq 1,\ = 0}$$

$$\geq \underbrace{\tfrac{1}{2}(-2K_X)(-2K_X - K_X)}_{= 0 \text{ b/c } K_X^2 = 0} + \cancel{\chi(\mathcal{O}_X)}\ 1$$

<u>Ex</u> if $h^0(X, \mathcal{O}_X(2K_X)) = 1 = h^0(X, \mathcal{O}_X(3K_X))$

$\Longrightarrow h^0(X, \mathcal{O}_X(K_X)) = 1$

$\Longrightarrow h^2(X, \mathcal{O}_X(3K_X)) = 0$

ii) $p_g = 0$ $q = 1$ $\chi(\mathcal{O}_X) = 0$

$\alpha : X \to E$ ← elliptic curve

run the same argument as in nonvanishing proof

$\Rightarrow$ α smooth fibration

$g(F) \geq 1$

<u>$g \geq 2$</u> $X' = E' \times F \to X$ étale $\kappa(X') = \kappa(E' \times F) \geq 1$

$g(F) \geq 2$ can't happen

canonical bundle

<u>$g = 1$</u> $\omega_X = \alpha^* \cancel{\omega_E}^{\mathcal{O}} \otimes \omega_{X/E} \overset{\checkmark}{=} \alpha^*(R^1\alpha_* \mathcal{O}_X)^\vee$

$= \alpha^* \alpha_* \omega_{X/E}$

$\alpha_* \omega_{X/E} \underset{?}{\cong} (R^1 \alpha_* \mathcal{O}_X)^\vee$

relative duality

but $\alpha_* \omega_{X/E}$ torsion bundle on E

$$\Rightarrow \quad (\alpha_* \omega_{X/E})^{\otimes m} = \mathcal{O}_E$$

$$\Rightarrow \quad \omega_X^{\otimes m} = \alpha^*(\alpha_* \omega_{X/E})^{\otimes m} = \alpha^* \mathcal{O}_E = \mathcal{O}_X$$

$m > 1$ b/c $p_g = 0$

iii) $P_g = 1 \quad g = 0 \qquad \chi(\mathcal{O}_X) = 2$

$$h^0(\mathcal{O}_X(-K_X)) + h^0(\mathcal{O}_X(2K_X))$$
$$\geq 0 + \chi(\mathcal{O}_X) = 2$$

$h^0(\mathcal{O}_X(2K_X)) \leq 1 \quad$ by $\quad \kappa(X) = 0$

$\Longrightarrow \quad h^0(\mathcal{O}_X(-K_X)) \geq 1$

$\Longrightarrow \quad K_X \sim 0$

iv) $P_g = q = 1 \qquad \chi(\mathcal{O}_X) = 1$

$h^1(X, \mathcal{O}_X) = \dim \operatorname{Pic}(X)$

$$\operatorname{Pic}^0(X) = H^1(X, \mathcal{O}_X) / \operatorname{im}(H^1(X, \mathbb{Z})) \neq 0$$

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$$

Pick some $M \in \operatorname{Pic}^0(X)$
2-torsion but not trivial

$h^0(\mathcal{O}_X(M)) = 0$

$h^2(\mathcal{O}_X(M)) = h^0(\mathcal{O}_X(K_X - M)) \quad 0$

$\geq \frac{1}{2} \cancel{M}(M-K) + \chi(\mathcal{O}_X) = 1$

$G \in |K_X - M| \Rightarrow 2G \in |2K_X - 2M|$
$= |2K_X|$

$D \in |K_X|$

$h^0(\mathcal{O}_X(2K_X)) \leq 1$ $=1$ $\quad |2K_X| \ni 2D = 2G$

$K_X \sim D = G \sim K_X - M$
$\Rightarrow M \sim 0$ contradiction

v) $p_g = 1 \quad q = 2 \quad \chi(\mathcal{O}_X) = 0$

$\alpha : X \to Alb(X) = A$ $\quad \dim A = 2$
abelian surface

<u>Claim 1</u>

α does not factor through a curve

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left, "\alpha"] & C \arrow[r] & A = Jac(C) \end{tikzcd}$$

$g(C) = 2$

Pick some $t': C' \to C$ finite étale cover with $g(C') > 2$

$$\begin{tikzcd} X \arrow[r] & C \\ X' \arrow[u] \arrow[r] & C' \arrow[u] \arrow[r] & \mathrm{Jac}(C') = \mathrm{Alb}(X') \end{tikzcd}$$

dim ≥ 3

$\kappa(X') = \kappa(X) = 0$

$q(X') \geq 3$ not possible

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so α is surjective

<u>Rieman-Hurwitz</u>

effective

$$K_X = \alpha^* K_A + E$$

Use intersection products + that $\exists$ component of E w/ neg self int

to conclude E contains a $g(C) = 1$ curve

$\implies$ have an elliptic curve on A

Complete reducibility theorem