Classification of surfaces

Thm (Abundance for surfaces)

Let X be a minimal model in dim 2

smooth projective surface w/ K_X nef

Then K_X is semiample.

Pf $\kappa(X) = 0$, $p_g = 1$ $q(X) = 2$ $\chi(\mathcal{O}_X) = 0$

$\alpha : X \to \mathrm{Alb}(X) = A$ ← 2-dimensional abelian variety

Claim α is surjective, & in particular generically finite.

Will conclude that α is étale

Riemann-Hurwitz

effective divisor measuring ramification

$$K_X = \alpha^* K_A + E = E$$

$K_A \sim 0$ since A abelian

If $E \neq 0$ then

Hodge index

$0 = K_X^2 = E^2 \Rightarrow \exists\ D \subseteq E$ an irreducible curve s.t. $\alpha(D) \neq pt$

if $\alpha(E)$ is zero dimensional, then by Hodge index $E^2<0$

$$E = n_0 D_0 + \sum_{i\geq 1} n_i D_i \qquad n_0 \geq 1$$

$$0 = K_X^2 = K_X . E \underset{\substack{\uparrow \\ K_X \text{ is nef}}}{\geq} \underline{K_X . n_0 D_0} = E . n_0 D_0 \geq n_0^2 D_0^2$$

$\Rightarrow$ $D_0^2 \leq 0$ $\qquad$ Adjunction $(K_X + D_0).D_0 \leq 0$

$\Rightarrow$ D_0 is rational or elliptic $\qquad$ $\deg K_{D_0} \leq 0$

if D_0 is rational, then Impossible

$$\begin{tikzcd} D_0 \arrow[r] \arrow[d] & X \arrow[r, "\alpha"] & A \\ \mathrm{Jac}(D_0) = pt \arrow[urr] & & \end{tikzcd}$$

but D_0 is chosen not contracted to a point

$\Rightarrow$ D_0 is an elliptic curve

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & A \arrow[d] \\ & A/\alpha(D_0) \end{tikzcd}$$

up to translating by some element of A, we can assume that $\alpha(D_0)$ is an abelian subvariety

$D_0 \subseteq \beta^{-1}(0)$ but $D_0^2 = 0$ $\Rightarrow$ $mD_0 = \beta^{-1}(0)$ by Hodge Index

$0 = \kappa(X) = \kappa(X, K_X) \geq \kappa(X, D_0) = 1$

b/c D_0 fiber of a map to a curve

Contradiction

$\Rightarrow$ E has empty

$0 = \alpha^* K_A = K_X = 0$ + α unramified

$\Rightarrow$ X is also an abelian variety ∎

<u>Thm</u> (Classification of Surfaces)

$K(X)$	$q(X)$	$P_g(X)$	$X^{min} = X$
$-\infty$	0	0	$\mathbb{P}^2$
$-\infty$	>0	0	$\mathbb{P}_C(\mathcal{E})$ $g(C) = q$ $\mathbb{P}^1 \times C$
0	0	1	K3 surfaces $K_X \sim 0$
0 a)	0	0	Enriques surfaces $2K_X \sim 0$, K3/involution
0	2	1	Abelian surfaces $K_X \sim 0$
0 b)	1	0	bielliptic surfaces $mK_X \sim 0$ $m > 1$ $E \times E'/G$ $\lvert G \rvert = m$
1 c)	≥ 0	≥ 0	Elliptic surface

2 ≥ 0 ≥ 0

Surfaces of general type K_X big + nef

resolutions of canonical models w/ $K_{X^{can}}$ ample

d)

Proof

a) + b)

$mK_X \sim 0$ for $m > 1$

$f \in H^0(X, \mathcal{O}_X(\underline{mK_X}))$ s.t. f is nonvanishing

Cyclic cover

$$\text{"}\{y^m = f\}\text{"} = \tilde{X} \xrightarrow{\beta} X$$

Idea is that on $\tilde{X}$, $K_{\tilde{X}}$ itself is trivial

$$\tilde{X} = \mathrm{Spec}_X \underbrace{\mathcal{O}_X \oplus \mathcal{O}_X(-K_X) \oplus \dots \oplus \mathcal{O}_X(-(m-1)K_X)}_{A}$$

$$\mathcal{O}_X(-aK_X) \otimes \mathcal{O}_X(-bK_X) \to \mathcal{O}_X(-(a+b-m)K_X)$$

$$(s, t) \longmapsto stf$$

$\beta: \tilde{X} \to X$ degree m cyclic cover

$X = \tilde{X}/\text{cyclic group}$

$K_{\tilde X} = \beta^* K_X = \mathcal{O}_{\tilde X}$ ← β is unramified

$$\beta_* \mathcal{O}_{\tilde X} = \mathcal{A} = \bigoplus_{a=0}^{m-1} \mathcal{O}_X(-aK_X)$$

$K(\tilde X) = K(X) = 0$ $p_g(\tilde X) = 1$

$$q(\tilde X) = h^1(\tilde X, \mathcal{O}_{\tilde X}) = h^1(X, \beta_* \mathcal{O}_{\tilde X})$$

$R^i \beta_* \cong 0$

$$= \sum_{a=0}^{m-1} h^1(X, \mathcal{O}_X(-aK_X))$$

a) $m = 2$ $\underset{0}{\underset{\|}{q(\tilde X)}} = \underbrace{\cancel{h^1(X, \mathcal{O}_X)}}_{0} + \underbrace{\cancel{h^1(X, \mathcal{O}_X(-K_X))}}_{0}$

$$\chi(\mathcal{O}_X(-K_X)) = \underbrace{\cancel{h^0(\mathcal{O}_X(-K_X))}}_{0} - h^1(X, \mathcal{O}_X(-K_X)) + \underbrace{\cancel{h^0(X, \mathcal{O}_X(2K_X))}}_{1}$$

$$1 - h^1(\mathcal{O}_X(-K_X)) = 0 + \chi(\mathcal{O}_X) = 1$$

$$h^1(\mathcal{O}_X(-K_X)) = 0$$

So $\tilde X$ $K_{\tilde X} \sim 0$ $q(\tilde X) = 0$

$\Longrightarrow X = \tilde X / \tau$ ← involution $\tilde X$ is a K3

b) bielliptic case

$$q(\tilde{X}) \geq \underset{\substack{\| \\ 1}}{q(X)} + h^1(X, \mathcal{O}_X(-(m-1)K_X))$$

$$h^1(X, \mathcal{O}_X(-(m-1)K_X)) = h^1(X, \mathcal{O}_X(mK_X))$$
$$= h^1(X, \mathcal{O}_X) = 1$$

$q(\tilde{X}) \geq 2$

by our classification, since $K_{\tilde{X}} \sim 0$

$\implies$ $q(\tilde{X}) = 2$ $\quad \tilde{X}$ abelian surface

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[rr, bend right] & X \arrow[r, "\alpha"] & \mathrm{Alb}(X) = E \end{tikzcd}$$
$\leftarrow$ elliptic curve

<u>Poincaré Reducibility Thm</u>

$E' \times E'' \to \tilde{X}$ (Splits up to Isogeny)

$$X = (E' \times E'')/G$$

<u>$K=1$</u> $\quad X$ minimal surface w/ $\kappa(X)=1$

$$\varphi_{|dK_X|} = \varphi : X \to C$$
$\leftarrow$ smooth curve

connected Fibers

F is a smooth fiber

$0 = F^2 \quad K_X.F = 0 \quad (K_X+F).F = K_F = 0$

$\Rightarrow$ F elliptic curve

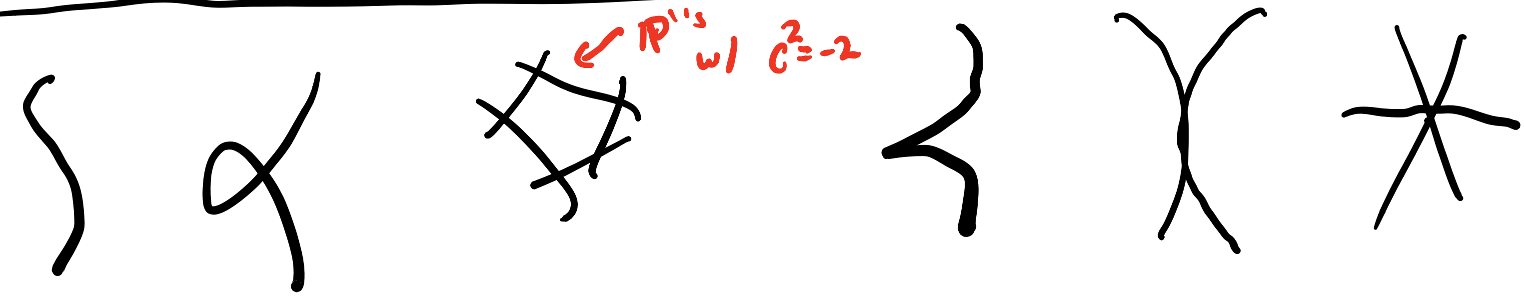
$\Rightarrow$ φ is an elliptic fibration

i.e. φ is a k-trivial fibration

$C^2=(-2)$

Classify Singular fibers

$\mathbb{P}^1$'s w/ $C^2=-2$



I_0 $\quad$ I_1 $\quad$ I_m $\quad$ II $\quad$ III $\quad$ IV

$m \geq 2$

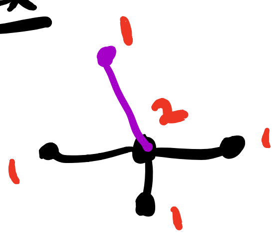
nonreduced

I_m^* $\quad$ II^* $\quad$ III^* $\quad$ IV^*

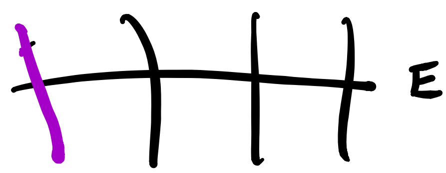
$\tilde{D}_{m+4}$ $\quad$ $\tilde{E}_8$ $\quad$ $\tilde{E}_7$ $\quad$ $\tilde{E}_6$

extended dynkin diagrams

Ex



$\tilde{D}_4$



A $\quad$ D_1 $\quad$ D_2 $\quad$ D_3

$C^2=(-2)$

C rational

A has mult one

$F = A + mE + n_1 D_1 + n_2 D_2 + n_3 D_3$

$F.C = 0 \quad C^2 = (-2)$ for any component

multiple fibers

$m I_n \qquad n \geq 0 \qquad m \geq 2$

no (-1) curves

Thm (Canonical bundle formula)

$\alpha : X \to C$ elliptic fibration relatively minimal

w/ multiple fibers $m_1 F_1, \dots, m_k F_k$.

then

$$\omega_X = \alpha^*\left(\omega_C \otimes (R^1 \alpha_* \mathcal{O}_X)^\vee\right) \otimes \mathcal{O}_X\left(\sum_{i=1}^{r} (m_i - 1) F_i\right)$$

$$K_X \sim_{\mathbb{Q}} \alpha^*(K_C + L + B)$$

$$B = \sum \frac{m_i - 1}{m_i} p_i \qquad m_i F_i = \alpha^{-1}(p_i)$$

$$\deg (R^1 \alpha_* \mathcal{O}_X)^\vee = \chi(\mathcal{O}_X)$$

Rmk • base of our K-trivial fibration is a pair $(C, L + B)$

$$\operatorname{Proj} R(K_X) = C = \operatorname{Proj} R(K_C + L + B)$$

geometry of X, $R(K_X)$ is ecoded in

C, $R(K_C + L + B)$

- L measured the "variation" of α

$$(R^1 \alpha_* \theta_X)^\vee = \alpha_* \omega_X$$

B measures singularities