$K(X)=1$

X is a minimal model of $k=1$, then

$$\varphi=\varphi_{|dK_X|}: X\to C$$ is a elliptic fibration

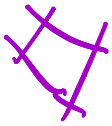
Step 1 classify fibers

$m\,I_d$

$m\,I_1$

$m\,I_0=$

α S



Step 2 Canonical bundle formula

Thm

$\alpha: X\to C$ be a relatively minimal elliptic fibration, with multiple fibers $m_1F_1, \ldots, m_nF_n$

$$\omega_X=\alpha^*\Big(\omega_C\otimes\underset{\underset{\mathcal{O}_C(L)}{\shortparallel}}{(R^1\alpha_*\mathcal{O}_X)^\vee}\Big)\otimes\underline{\mathcal{O}_X\Big(\sum_i(m_i-1)F_i\Big)}$$

$$K_X\sim_{\mathbb{Q}}\alpha^*(K_C+L+B)$$

$$B=\sum\frac{m_i-1}{m_i}p_i \qquad m_iF_i=\alpha^{-1}(p_i)$$

$$+\quad \deg(R^1\alpha_*\mathcal{O}_X)^\vee=\chi(\mathcal{O}_X)$$

PF sketch

$$\mathcal{O}_F(F)^{\otimes m} = \mathcal{O}_F(mF) = \mathcal{O}_X(\alpha^* p)|_F = \mathcal{O}_F$$

Lemma 1:

1) $\alpha^{-1}(p) = mF$, then

$\omega_F = \mathcal{O}_F$ $\quad$ $\mathcal{O}_F(F)$ is a torsion bundle of order m

2) $h^0(\mathcal{O}_{\alpha^{-1}(p)}) = h^1(\mathcal{O}_{\alpha^{-1}(p)}) = 1$

for <u>all</u> $p \in C$

$$h^0(F, \mathcal{O}_X(K_X)|_F) \underset{\text{adjunction}}{=} h^0(F, \omega_F) \underset{\text{Serre duality}}{=} \overset{=1}{h^1(F, \mathcal{O}_F)}$$

for <u>all</u> fibers

<u>Grauert's thm</u> $\Rightarrow$ $\alpha_* \mathcal{O}_X(K_X)$ is a line bundle

$$\alpha^* \alpha_* \mathcal{O}_X(K_X) \xrightarrow{\lambda} \mathcal{O}_X(K_X)$$

Claim 1 x is an isomorphism on the generic fiber X_η of α

$U \subseteq X$ locus where α is smooth

$$\omega_U = \omega_X\big|_U = (\alpha^*\omega_C \otimes \omega_{X/C})\big|_U = \alpha^*\omega_{\alpha(U)} \otimes \omega_{U/\alpha(U)}$$

restrict to X_η

$$\omega_{X_\eta} = \mathcal{O}_{X_\eta}$$

$$\Rightarrow \quad \mathcal{O}_X(K_X) = \alpha^*\alpha_*\mathcal{O}_X(K_X) \otimes \mathcal{O}_X(D)$$

D effective and contained in fibers

$$0 = K_X^2 = (\alpha^* B + D)^2 = D^2$$

$\boxed{m_i \geq n_i > 0.}$ Exc

$$\Rightarrow \quad D = \sum n_i \underbrace{\alpha^{-1}(p_i)}_{\substack{\shortparallel \\ F_i}}$$ for some n

$$\mathcal{O}_{F_i} = \omega_{F_i} = \mathcal{O}_X(K_X + F_i)\big|_{F_i} = \mathcal{O}_X(n_iF_i + F_i)\big|_{F_i}$$

$$= \mathcal{O}_F((n_i+1)F_i)$$

$n_i + 1 = m_i \qquad n_i = m_i - 1$

Next

$$\alpha_* \mathcal{O}_X(K_X) = \alpha_*(\alpha^* \omega_C \otimes \omega_{X/C})$$

$$= \omega_C \otimes \alpha_* \omega_{X/C}$$

$$= \omega_C \otimes (R^1\alpha_* \mathcal{O}_X)^\vee$$

Relative Duality

$$\alpha_* \omega_{X/C} = (R^1\alpha_* \mathcal{O}_X)^\vee$$

Use RR to compute + Leray spectral sequence

$$\chi(\mathcal{O}_X) = -\deg R^1\alpha_* \mathcal{O}_X$$

$K=2$

X is a minimal model of general typ

by Abundance, we know

$$\varphi = \varphi_{|dK_X|} : X \to Z \qquad \dim Z = 2$$

$$= \operatorname{Proj} R(K_X)$$

$E \subseteq \operatorname{Exc}(\varphi)$

$$E^2 < 0$$

$$K_X . E = 0$$

$K_E = (K_X + E).E < 0 \Rightarrow E \cong \mathbb{P}^1$

$E^2 = -2$

$\Rightarrow$ $\mathrm{Exc}(\varphi) =$ disjoint union of trees of (-2) curves w/ negative definite intersection matrix

Classified by Dynkin diagrams of type ADE

Canonical singularities

$\Rightarrow$ Z has ADE singularities at points where some (-2) curves are contracted

Low degree exact seq of Leray SS

$$0 \to \mathrm{Pic}(Z) \xrightarrow{\varphi^*} \mathrm{Pic}(X) \to \bigoplus_{E \in \mathrm{Exc}} \mathbb{Z}$$

Compute this using that $\mathrm{Pic} = H^1(X, \mathcal{O}_X^*)$ + $\varphi_* \mathcal{O}_X = \mathcal{O}_Z$

$K_X = \varphi^* L$ $\qquad$ $L|_{Z^{sm}} = K_{Z^{sm}}$

$L \in \mathrm{Pic}(Z)$ $\qquad$ $\Rightarrow L = K_Z$

$K_X = \varphi^* K_Z$ & so K_Z is ample if $C \not\subseteq \mathrm{Exc}(\varphi)$ on X $\Uparrow$

$\Rightarrow K_X . C > 0$

$\Rightarrow K_Z . C' > 0$ for any $C' \subseteq Z$

$$\varphi : X^{min} \longrightarrow Z = X^{can} = \mathrm{Proj}\, R(K_X)$$

uniquely characterized by

1) having canonical sing

2) K_Z is ample

Goal 1) Cone + contraction theorems
+ generalize to singularities

2) Base point free theorem, rationality, + Vanishing theorems
+ non-vanishing

The cone of curves in higher dimension

X projective

D cartier divisor, $C \overset{i}{\subseteq} X$ a curve

$$D . C = \deg i^* \mathcal{O}_X(D)$$

$C = \sum a_i C_i$ 1-cycle, extend by linearity

$C \equiv C'$ iff $D . C = D . C'$ for all Cartier D

$N_1(X) := N_1(X)_{\mathbb{R}}$ = numerical eq classes of $\mathbb{R}$-1-cycle

$a_i \in \mathbb{R}$

$N^1(X) = NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$

$NS(X) = \mathrm{Pic}(X) / \mathrm{Pic}^{\circ}(X)$ = divisors up to alg eq

$N_1(X) \otimes N^1(X) \longrightarrow \mathbb{R}$ perfect pairing

$\rho(X) = \dim N_1(X) < \infty$ for any X

X smooth, $N_1(X) \hookrightarrow H_2(X, \mathbb{R})$

$\le b_2(X)$

Def

$$NE_{\mathbb{Q}}(X) = \left\{ \sum a_i [C_i] \;\middle|\; 0 \le a_i \in \mathbb{Q} \right\}$$

$$\cap$$

$$NE(X) = \left\{ \quad '' \quad \middle|\; 0 \le a_i \in \mathbb{R} \right\}$$

$$\cap$$

$\overline{NE}(X)$ = closure inside $\subseteq N_1(X)$

D cartier

$\mathbb{Q}$-Cartier

$\mathbb{R}$-Cartier

$$D_{\geq 0} := \{ x \in N_1(X) \mid D . x \geq 0 \}$$

$>$

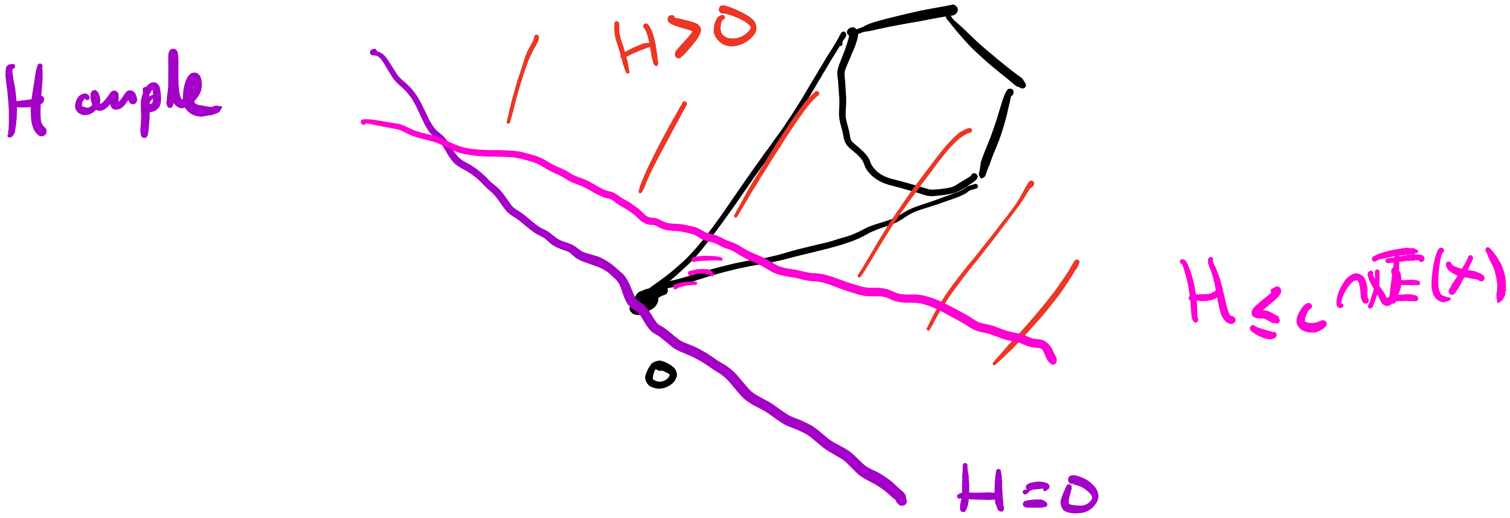
$\le$

$<$

$$\overline{NE}(X)_{\substack{D\geq 0\\ >\\ \leq\\ <}} := \overline{NE}(X)\cap D_{\substack{\geq 0\\ >\\ \leq\\ <}}$$

Thm (Kleiman's criterion) If X is projective, D cartier, then D is ample iff

$$D_{>0} \supseteq \overline{NE}(X) \setminus \{0\}$$



Cor 1) $\overline{NE}(X)$ doesn't contain a line

2) $\{z \mid H.z \leq c\}$ for any fixed c is compact

3) the set of integral 1-cycles $\overline{NE}(X) \ni Z = \sum n_i [C_i]$ w/ $H.Z \leq c$ is finite

Thm (Cone Theorem I)

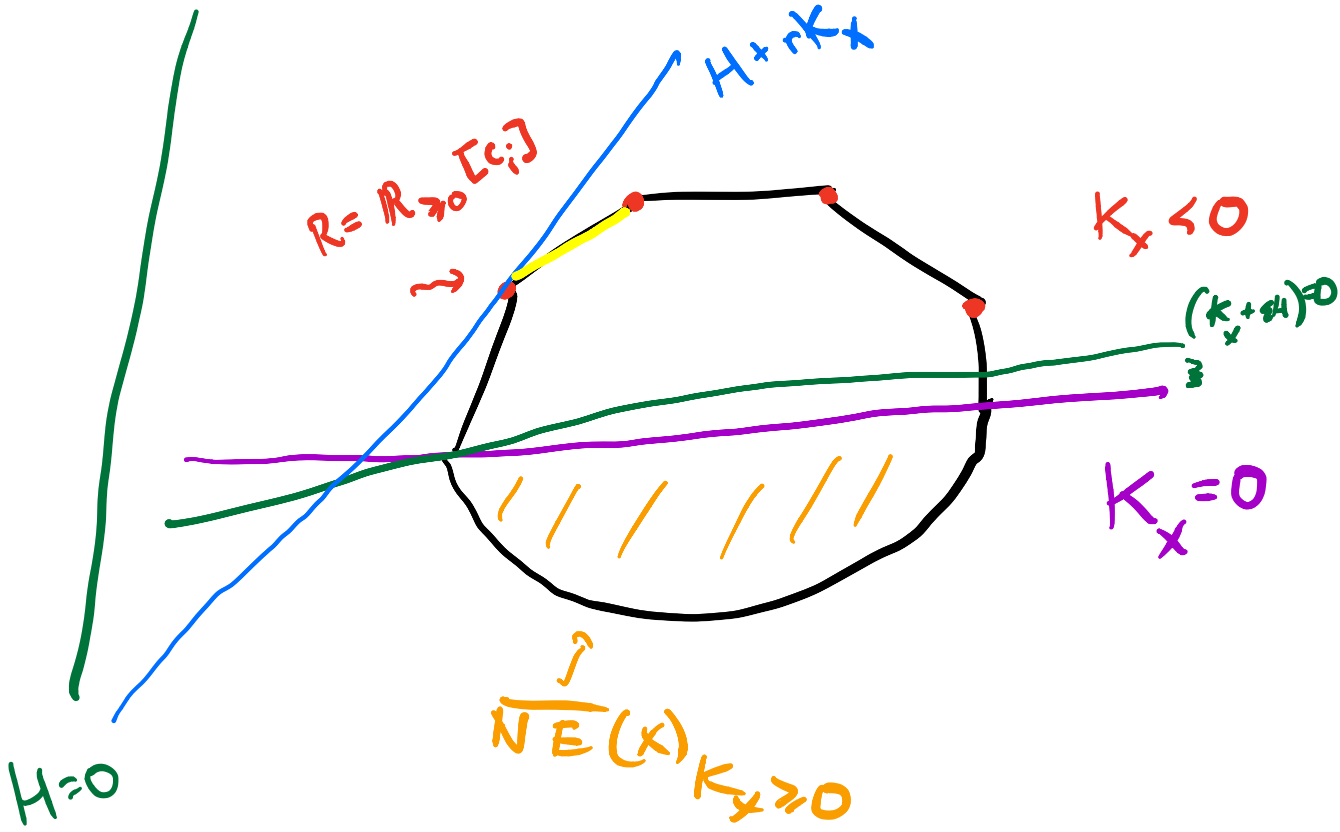
let X be a smooth projective variety. Then there exist countably many rational curves C_i such that

1) $\overline{NE}(X) = \overline{NE}(X)_{K_X \geq 0} + \sum_i \mathbb{R}_{\geq 0}[C_i]$

where $0 < -(K_X \cdot C_i) \leq \dim X + 1$

2) For any ample H, $\varepsilon > 0$

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X + \varepsilon H) \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_i]$$



Rmk $R = \mathbb{R}_{\geq 0}[C_i]$ are extremal rays

meaning if $v, w \in \overline{NE}(X)$

s.t. $v + w \in R \Rightarrow v, w \in R$

Def $F \subseteq \overline{NE}(X)$ any extremal face (K_X - negative)

a morphism $\mathrm{cont}_F : X \to Y$ is an <u>extremal contraction of F</u> if

1) $\mathrm{cont}_F(C) = pt \iff [C] \in F$
2) $\mathrm{cont}_{F*} \mathcal{O}_X = \mathcal{O}_Y$

Thm (Contraction Theorem I)

let R is a K_X-negative extremal ray, then there is <u>unique</u> extremal contraction to a projective Y

$\mathrm{cont}_R : X \to Y$, and

if L a line bundle on X w/ $L . C = 0$ for all $[C] \in R$

then $\exists$ L_Y on Y s.t. $\mathrm{cont}_R^* L_Y = L$
line bundle