Extremal contractions

$R \subseteq \overline{NE}(X)$ extremal, $K_X < 0$

$f = \mathrm{cont}_R : X \to Y$ 1) $f_* \mathcal{O}_X = \mathcal{O}_Y$

2) C is contracted by f $\iff$ $[C] \in R$

Def-Prop an extremal contraction f (K_X is $\mathbb{Q}$-cartier) (X is $\mathbb{Q}$-factorial) is one of the following

1) Fiber space ($\dim X > \dim Y$)

2) divisorial contraction (f is birational & $Ex(f)$ irreducible divisor)

3) small contraction (f birational but $Ex(f)$ has codim ≥ 2)

Proof The only statement requiring proof is that if $E \subseteq Ex(f)$ is an irreducible divisor, then $E = Ex(f)$

$E \cdot R < 0$ where R is our extremal ray

(higher dim analogue of Hodge index)

OTOH, $C \subseteq Ex(f) \Rightarrow [C] \in R$ so

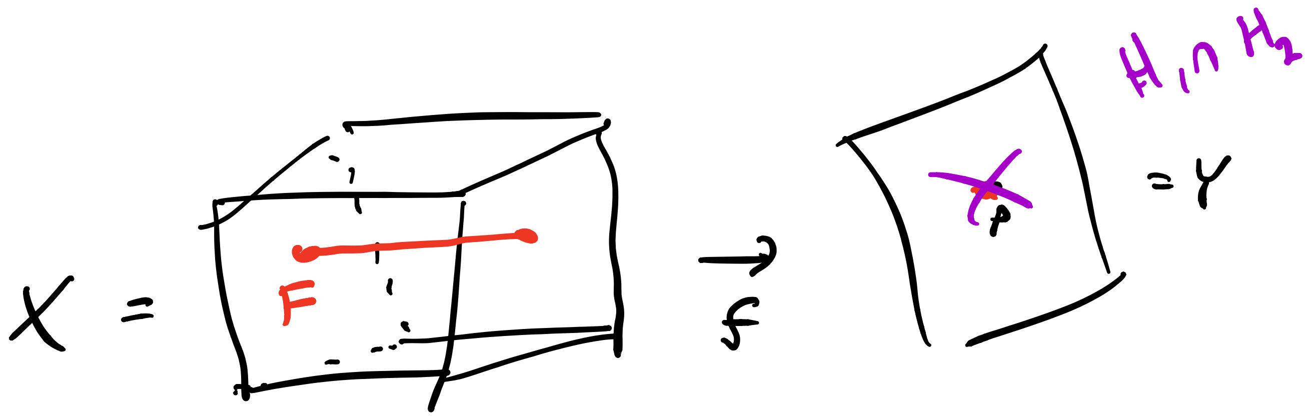
$E.C < 0 \Rightarrow C \subseteq E \Rightarrow E = Ex(f)$

)) $f : X \to Y$ with connected fibers

F general fiber

$-K_X|_F$ ample by Kleiman's criterion

$$K_F = K_X|_F$$



$f^*H_i = D_i$, $F = D_1 \cap D_2$, $D_i|F = 0$

$$((K_X + D_1 + D_2)|_{D_1})|_{D_2|_{D_1}} = (K_{D_1} + D_2|_{D_1})|_{D_2|_{D_1}} = K_{D_1 \cap D_2} = K_F$$

$$= (K_X + D_1 + D_2)|_F = K_X|_F$$

$\Rightarrow$ general fiber is Fano
+ all fibers are irreducible
(Mori fiber space)

2) divisorial contraction

$\rho(X) = \rho(Y) + 1$

Need to know that singularities of Y are nice enough to continue
(ideally, we want Y to be $\mathbb{Q}$-fact)

3) small contraction, $\rho(X)$ doesn't drop
K_Y can't be $\mathbb{Q}$-Cartier!

1) $K_X \cdot C < 0$ for $C \subseteq$ fibers of f

Suppose K_Y $\mathbb{Q}$-Cartier

$f^* K_Y = K_X$

$K_X \cdot C = f^* K_Y \cdot C =$
$K_Y \cdot f_* C = 0$

so K_Y can't be $\mathbb{Q}$-Cartier

Def f small extremal contraction of $K_X<0$ ray
a flip of f is a birational
morphism $f_+ : X_+ \to Y$

$$\begin{tikzcd} X \arrow[rr, dashed, leftrightarrow] \arrow[dr, "f"'] & & X_+ \arrow[dl, "f_+"] \\ & Y & \end{tikzcd}$$

$-K_X$ f-ample

K_{X_+} is f_+-ample

1) f_+ is a small contraction (codim Ex ≥ 2)

2) K_{X_+} is $\mathbb{Q}$-Cartier

3) $K_{X_+} \cdot C > 0$ for $C \subseteq Ex(f_+)$

big Question: when do flips exist?

Outline of minimal model program

1) identify some class of singularities such that cone + contraction holds, & they're closed under divisorial contraction + flips

2) Need existence of flips

3)

$$\begin{tikzcd} X \arrow[r, "\text{contr}"] & X_1 \arrow[r, dashed] & X_2 \arrow[r, dashed] & X_m \arrow[dr] & \\ & & & & Z \end{tikzcd}$$

if divisorial $X = X_1$

is small replace X_1 by a flip $X_1^+ = X$

if fiber space then we've reduced to lower dimensions

K_{X_m} nef

or

Mori fiber space

BCHM under bigness assumption $\Rightarrow \exists$ an MMP terminating in

4) termination of flips? a minimal model

5) Need to understand X_m

<u>Conj</u> if X_m is a minimal model then K_{X_m} is semi-ample

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Positivity for divisors

H ample line bundle

$\phi_{|mH|} : X \hookrightarrow \mathbb{P}^N$ $W \subseteq X$ any subvariety

$$H^{\dim W} . W = \frac{1}{m} \deg (\overset{\phi}{W} \subseteq \mathbb{P}^N)$$

<u>Lemma</u> D effective & H ample then $D + mH$ ample for any $m \gg 0$

$\Longrightarrow$ $D^{\dim W} . W$ by linearity

<u>Thm</u> (Nakai-Moishezon Theorem) X is proj D cartier, then D is ample

$\Longleftrightarrow$ $D^{\dim W} . W > 0$ for all W

Pf $\Rightarrow$: clear

$\Leftarrow$: May suppose X is irreducible

Induct on $\dim X = n-1$

$D|_Z$ ample for all $Z \subsetneq X$

Step 1 $h^0(\mathcal{O}_X(mD)) > 0$ for $m \gg 0$

D is big

Pick $D+H$ is very ample

$A \in |D+H|$

$$0 \to \mathcal{O}_X(kD-H) \to \mathcal{O}_X(kD) \to \mathcal{O}_H(kD) \to 0$$
(*)

$$0 \to \mathcal{O}_X(kD-H) \to \mathcal{O}_X((k+1)D) \to \mathcal{O}_A(kD) \to 0$$
(**)

(*) $h^i(\mathcal{O}_H(kD)) = 0$ for $k \gg 0$ $i \geq 1$

$$h^i(\mathcal{O}_X(kD-H)) = h^i(\mathcal{O}_X(kD)) \quad i \geq 2$$
$$\|$$

(**) $h^i(\mathcal{O}_X(kD-H)) = h^i(\mathcal{O}_X((k+1)D))$ $i \geq 2$

$$h^0(\mathcal{O}_X(kD)) \geq h^0(\mathcal{O}_X(kD)) - h^1(\mathcal{O}_X(kD))$$
$$= \chi(\mathcal{O}_X(kD)) + \text{constant}$$
$$= \frac{D^n k^n}{n!} + \text{lower order}$$
>0

<u>Thm</u> (very weak Riemann-Roch) $n = \dim X$

$$\chi(\mathcal{O}_X(tD)) = \frac{D^n}{n!} t^n + \text{lower order in } t$$

<u>Step 2</u> Assume D effective by step 1

Want $\mathcal{O}_X(kD)$ is globally generated

e.g. $|kD|$ is bpf $k \gg 0$

$s \in |D|$

$$0 \to \mathcal{O}_X((k-1)D) \to \mathcal{O}_X(kD) \to \mathcal{O}_s(kD) \to 0$$

globally gen by induction

So it suffices to show

$H^0(\mathcal{O}_X(kD)) \twoheadrightarrow H^0(\mathcal{O}_s(kD))$ for $k \gg 0$

$$h^1(\mathcal{O}_S(kD)) = 0 \implies$$

$$H^1(\mathcal{O}_X((k-1)D)) \twoheadrightarrow H^1(\mathcal{O}_X(kD))$$

$$\twoheadrightarrow H^1((k+1)D)$$

$\rightsquigarrow$ surjectivity $\Rightarrow$ eventually isomorphism

$\Longrightarrow$ restriction is surjective

Step 3 by step 2, $\varphi = \varphi_{|kD|} : X \to \mathbb{P}^n$

$kD = \varphi^* H$, but φ is finite

$0 < D.C = H.\varphi(C) \Rightarrow \varphi|_C$ is finite

Prop $\varphi : X \to Y$ finite, L ample on $Y \Rightarrow \varphi^* L$ ample on X

Pf

$$H^i(\mathcal{F}(m\varphi^* L)) \overset{\text{finiteness}}{=} H^i(\varphi_*(\mathcal{F} \otimes m\varphi^* L))$$

$$= H^i(\varphi_* \mathcal{F}(mL))$$

$$= 0 \quad b/c \quad L \text{ ample}$$

☺

Cor being ample is numerical

$D_1 \equiv D_2$ & D_1 ample $\Rightarrow D_2$ ample

Thm X proj L nef $\Rightarrow$

$$L^{\dim Z} \cdot Z \geq 0 \quad \text{for } Z \subseteq X$$

Pf Induct on dimension, $\Rightarrow$

$$L^{\dim Z} \cdot Z \quad \text{for all } Z \not\subseteq X$$

just need that $L^{\dim X} \geq 0$ $\quad n = \dim X$

A very ample

$$f(x,y) = (xL + yA)^n = x^n L^n + \sum \underbrace{L^{n-i} A^i}_{\geq 0} x^{n-i} y^i \binom{n}{i} + y^n \underbrace{A^n}_{>0}$$

$f(1,t) \Rightarrow$ increasing function of t

Suppose $L^n < 0$ $\quad \exists\, t_0 > 0$

$= f(1,0)$ $\quad$ with $f(1, t_0) = 0$

Pick $\mathbb{Q} \ni t_1 > t_0$

$(L + t_1 A)^{\dim Z} Z > 0 \quad Z \not\subseteq X$

by induction

$(L + t_1 A)^{\dim X} = f(1, t_1) > 0$

$\Rightarrow (L + t_1 A)$ ample by NM

$$0 = (L + t_0 A)^n = L \cdot (L + t_0 A)^{n-1} + t_0 (A \cdot (L + t_0 A)^{n-1})$$

$(L + t_0 A)^{n-1} = \lim_{t_1 \to t_0} (L + t_1 A)^{n-1} \geq 0$, $\quad t_0 (A \cdot (L + t_0 A)^{n-1}) > 0$

Contradiction

$\Rightarrow L^n \geq 0$ ∎

<u>Cor</u> (Kleiman's criteria) D ample

$$\Longleftrightarrow \quad D_{>0} \supseteq \overline{NE}(X) \setminus \{0\}$$

<u>Pf</u> $\Rightarrow$ D ample, pick a basis for $N^1(X)$ $D = D_1, \dots, D_{\rho(X)}$

s.t. D_i are ample, $2D - D_i$ are ample

$\|x\| = \sum |(D_i \cdot x)|$ for $x \in N_1(X)$

$z \in \overline{NE}(X) \setminus \{0\}$

$$2\rho(X)\, D.z - \|z\| =$$

$$2\rho(X)\, D.z - \sum_{i=1}^{\rho(X)} D_i . z$$

$$= \rho(X) \left(\sum (2D - D_i).z \right) \geq 0$$

$$2\rho(X)\, D.z \geq \|z\| > 0$$

$$\Rightarrow D.z > 0$$

$\Leftarrow$: Suppose $D.z > 0$ for

$z \in \overline{NE}(X) \setminus \{0\}$

Pick A ample, pick t

$L = tD - A$ is Nef

$$tD = L + A$$

$$(tD)^{\dim Z} . Z = (L+A)^{\dim Z} . Z > 0$$

≥ 0 $\quad$ > 0

by previous theorem

$\Rightarrow$ D ample by NM