Math 290 pset 1

Due 11/23/2020

1. Show that the canonical ring
$$R(K_X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$
is a birational invariant for smooth projective X. Conclude the same for X with at worst canonical singulartiies.

2. Let $f : X \to Y$ be a Fano fibration, that is, a morphism between normal varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and such that the generic fiber of f is Fano. Suppose that X and Y have log canonical singularities. Show that $\kappa(X) = -\infty$.

3. Let X be a smooth variety, $Z \subset X$ a smooth subvariety and $\Delta = \sum a_i D_i$ a simple normal crossings divisor. Let $f : Y = \mathrm{Bl}_Z(X) \to X$ be the blowup of X along Z with exceptional divisor E. Show that the discrepency of E over (X, D) can be computed as
$$a(E, X, \Delta) = k - 1 - \sum a_i \mathrm{mult}_Z(D_i).$$

4. Let $f : Y \to X$ be a proper birational morphism and Δ_X and Δ_Y $\mathbb{Q}$-divisors such that
   (a) (X, Δ_X) and (Y, Δ_Y) are log pairs,
   (b) $f_*\Delta_X = \Delta_Y$, and
   (c) $f^*(K_X + \Delta_X) = K_Y + \Delta_Y$.

   Show that for any prime divisor P lying over X, we have
$$a(P, X, \Delta_X) = a(P, Y, \Delta_Y).$$

5. Prove the following relative Kawamata-Viehweg vanishing theorem. Suppose $f : (X, \Delta) \to Z$ is a morphism between projective varieties such that (X, Δ) is klt and Δ is effective. Let L be a Cartier divisor on X with $L \equiv M + \Delta$ for M an f-big and f-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Show that $R^i f_* \mathcal{O}_X(K_X + L) = 0$ for $i > 0$.

   You may need to use the usual version of KV vanishing below.

   Theorem 1. *(Kawamata-Viehweg vanishing) Let (X, Δ) be a projective klt pair with Δ effective. Suppose L is a Cartier divisor with*
$$L \equiv M + \Delta$$
   for M a big and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then
$$H^i(X, \mathcal{O}_X(K_X + L)) = 0 \qquad i > 0.$$

6. Let (X, D) be a smooth variety with smooth divisor $D = V(s)$ for $s \in H^0(X, \mathcal{L}^m)$ for some line bundle $\mathcal{L}$ and let

$$p : X' = X_{m,D} \to X$$

be the associated m-fold cyclic cover defined by

$$X_{m,D} = \mathrm{Spec}_X \bigoplus_{k=0}^{m-1} \mathcal{L}^{-k}$$

where the algebra structure is given by the composition

$$\mathcal{L}^{-a} \otimes \mathcal{L}^{-b} \to \mathcal{L}^{-a-b} \xrightarrow{\cdot s} \mathcal{L}^{m-a-b}.$$

(a) Show that X' is smooth and $p^*D = mD'$ where D' is a smooth divisor.

(b) Define $\Omega^1_X(\log D)$, the sheaf of differentials with logarithmic poles along D, is the subsheaf of the sheaf of meromorphic differentials locally spanned by

$$\frac{dx_1}{x_1}, dx_2, \ldots, dx_n$$

whenever D is locally given as the vanishing locus $x_1 = 0$. Show that there exist exact sequences

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \xrightarrow{\alpha} \mathcal{O}_D \to 0$$
$$0 \to \Omega^1_X(\log \mathrm{D})(-D) \to \Omega^1_X \xrightarrow{\beta} \Omega^1_D \to 0$$

where α is the residue map $\omega \mapsto \mathrm{Res}_D(\omega)$ and β is the restriction to D.

(c) Show the following version of the Riemann-Hurwitz formula:

$$p^*\Omega^p_X(\log D) = \Omega^p_{X'}(\log D').$$

Recall that $\Omega^p_X(\log D) := \wedge^p \Omega^1_X(\log D)$. where

(d) Use the the sequences below (generalizing those proved in part (b))

$$0 \to \Omega^p_X \to \Omega^p_X(\log D) \to \Omega^{p-1}_D \to 0$$
$$0 \to \Omega^p_X(\log D)(-D) \to \Omega^p_X \to \Omega^p_D \to 0$$

to complete the proof of Kodaira-Akizuki-Nakano vanishing as outlined in class.

Theorem 2. *(Kodaira-Akizuki-Nakano vanishing) Let X be a smooth projective variety and $\mathcal{L}$ an ample line bundle. Then*

$$H^q(X, \Omega^p_X \otimes \mathcal{L}^{-1}) = 0$$

for $p + q < n$.