MATH 445 – Exam II – 6 November 2009

Work each numbered problem on a separate answer sheet. Show all your work for each problem clearly on the answer sheet for that problem. Write your name and the problem number on each answer sheet. Good luck.

1. [10 pts] (a) Let $\mathcal{L}^{nl} = \{R, E, O, P\}$ where R is a binary relation symbol and E, O, P are unary relation symbols. Let $\mathcal{A}$ be the $\mathcal{L}$-structure with universe $\mathbb{N}$ which interprets R by $<$, E by the set of even numbers, O by the set of odd numbers, and P by the set of prime numbers. Give sentences of $\mathcal{L}$ which "naturally" express the following facts about $\mathcal{A}$.

(i) Every odd prime is bigger than any even prime.

(ii) The immediate successor of any even number is odd.

[15 pts] (b) Let $\mathcal{L}^{nl} = \{R\}$ where R is a binary relation. Let $\mathcal{A}$ be the $\mathcal{L}$-structure $(\mathbb{Z}, <)$. Give a formula $\psi(x, y)$ such that $\mathcal{A} \models \psi(k, l)$ iff $k = l - 2$. Explain.

2. Working directly from the definitions, either prove or find a counterexample to each of the following. Explain carefully.

[15 pts] (a) $\models (\exists x P(x) \to \forall x Q(x)) \to \forall x (P(x) \to Q(x))$

[15 pts] (b) $\models (\forall x P(x) \to \forall x Q(x)) \to \forall x (P(x) \to Q(x))$

3. [15 pts] Using the rules in the notes (but not, of course, the Completeness Theorem) show that

$\vdash (\forall x P(x) \to \forall x Q(x)) \to \exists x (P(x) \to Q(x))$

4. [15 pts] Either prove (directly from the definitions) or disprove (with a counterexample) each of the following:

(a) $\forall x \neg \varphi(x)$ is satisfiable iff $\forall x \varphi(x)$ is not satisfiable.

(b) If either $\models \forall x (\varphi(x) \to \theta(x))$ or $\models \forall x (\psi(x) \to \theta(x))$ then $\models \forall x \big((\varphi(x) \wedge \psi(x)) \to \theta(x)\big)$.

5. [5 pts] (a) Define what it means for a set Σ of sentences of $\mathcal{L}$ to be *maximal consistent*.

[10 pts] (b) Let Σ be a maximal consistent set of sentences of $\mathcal{L}$. Prove that for every $\theta \in Sn_{\mathcal{L}}$, $\theta \in \Sigma$ iff $\Sigma \vdash \theta$. (Do not assume the Completeness Theorem).

NOTE: Your solutions must include enough detail to justify your conclusions.