MATH 445 – Exam II – 5 November 2010

Work each numbered problem on a separate answer sheet. Show all your work for each problem clearly on the answer sheet for that problem. Write your name and the problem number on each answer sheet. Good luck.

1. [10 pts] (a) Let $\mathcal{L}^{nl} = \{R, E, O\}$ where R is a binary relation symbol and E, O are unary relation symbols. Let $\mathcal{A}$ be the $\mathcal{L}$-structure with universe the set of positive integers and which interprets R by $|$ (divides), E by the set of even positive integers, and O by the set of odd positive integers. Give sentences of $\mathcal{L}$ which "naturally" express the following facts about $\mathcal{A}$.

(i) Every odd number divides some even number.

(ii) Some even numbers have no odd divisors.

[15 pts] (b) Let $\mathcal{L}^{nl} = \{F\}$ where F is a binary function symbol. Let $\mathcal{A}$ be the $\mathcal{L}$-structure $(\mathbb{Z}, +)$ and let $\mathcal{B}$ be the $\mathcal{L}$-structure $(\mathbb{Q}, +)$. Give a sentence σ of $\mathcal{L}$ such that $\mathcal{A} \models \sigma$ but $\mathcal{B} \models \neg\sigma$. Explain.

2. Working directly from the definitions, either prove or find a counterexample to each of the following. Explain carefully.

[15 pts] (a) $\models \exists x(P(x) \to Q(x)) \to (\exists x P(x) \to \exists x Q(x))$

[15 pts] (b) $\models (\forall x P(x) \to \exists x Q(x)) \to \exists x(P(x) \to Q(x))$

3. [15 pts] Using the rules in the notes (but not, of course, the Completeness Theorem) show that

$\vdash \forall x(P(x) \to Q(x)) \to (\exists x P(x) \to \exists x Q(x))$

4. [15 pts] Either prove (directly from the definitions) or disprove (with a counterexample) each of the following:

(a) If $\exists x\varphi(x)$ and $\exists x\psi(x)$ are both valid then so is $\exists x(\varphi(x) \wedge \psi(x))$.

(b) $\forall x\varphi(x)$ is valid iff $\exists x\neg\varphi(x)$ is not satisfiable.

5. [5 pts] (a) Define what it means for a set Σ of sentences of $\mathcal{L}$ to be *maximal consistent*.

[10 pts] (b) Let Σ be a maximal consistent set of sentences of $\mathcal{L}$. Assume that $\forall x P(x) \in \Sigma$. Prove that $P(c) \in \Sigma$ for every constant $c \in \mathcal{L}$. (Do not assume the Completeness Theorem).

NOTE: Your solutions must include enough detail to justify your conclusions.