MATH 445 — Exam 2 Review

Topics

- first order languages and structures for first order languages
- terms, atomic formulas, formulas, free/bound occurrences of variables, sentences
- interpreting terms and formulas in structures, the satisfaction relation $\mathcal{A} \models \varphi(a_1, \ldots, a_n)$, the truth relation $\mathcal{A} \models \theta$
- validity, satisfiability, and logical consequence
- techniques for determining validity, satisfiability, and logical consequence
- properties of validity and logical consequence
- logical axioms, MP, deductions, deductions from hypotheses, Soundness
- the Deduction Theorem, Generalization, other derived rules, and their use in showing deducibility
- expressability in first order logic
- consistent sets of sentences, maximal consistent sets
- extending consistent sets to maximal consistent sets
- adding "witnesses" to consistent sets
- defining a structure from a maximal consistent set with witnesses
- Model Existence Theorem, Completeness Theorem (both forms)

Sample Problems

1. Let φ be the formula $\big(\forall x(R(x,y) \to \exists y R(y,y)) \vee (\exists z R(z,x) \to \forall x R(y,z))\big)$. List all subformulas of φ and determine which ocurrences of variables in φ are free and which are bound.

2. Let $\mathcal{L}^{nl}$ contain just the binary relation symbol R. Let $\mathcal{A} = (\mathbb{N}, \leq)$. Determine the sets defined in $\mathcal{A}$ by the following formulas.

(a) $\varphi(x)$ is $\forall y(R(y,x) \to \neg R(y,x))$

(b) $\psi(x)$ is $\forall y(R(y,x) \to R(x,y)$

(c) $\theta(x)$ is $\forall y(R(x,y) \to R(y,x))$

3. Let $\mathcal{L}$ be as in problem 2. Let $\mathcal{A}$ be the $\mathcal{L}$-structure $(\mathbb{N}, \leq)$ and let $\mathcal{B}$ be the $\mathcal{L}$-structure $(\mathbb{N}, |)$, where $k|n$ iff k divides n. Find a sentence σ of $\mathcal{L}$ such that $\mathcal{A} \models \sigma$ but $\mathcal{B} \models \neg\sigma$.

4. Working directly from the definitions, either prove or find a counterexample to each of the following.

(a) $\models (\forall x P(x) \to \forall x Q(x)) \to \forall x(P(x) \to Q(x))$

(b) $\models (\forall x P(x) \to \forall x Q(x)) \to \exists x(P(x) \to Q(x))$

5. Prove or disprove (with a counterexample) each of the following:

(a) If the sentences $\exists x\varphi(x)$ and $\exists x\psi(x)$ are both satisfiable then so is $\exists x\big(\varphi(x) \wedge \psi(x)\big)$.

(b) If $\models \forall x(\varphi(x) \to \psi(x))$ then $\exists x\psi(x)$ is satisfiable.

(c) If $\models \forall x(\varphi(x) \to \theta(x))$ and $\models \forall x(\psi(x) \to \theta(x))$ then $\models \forall x((\varphi(x) \vee \psi(x)) \to \theta(x))$.

6. Establish the following, using the rules in the notes.

(a) $\vdash \forall x(P(x) \to Q(x)) \to (\forall x P(x) \to \forall x Q(x))$

(b) $\vdash (\forall x P(x) \to \exists x Q(x)) \to \exists x(P(x) \to Q(x))$

7. (a) Define what it means for a set Γ of sentences of $\mathcal{L}$ to be *maximal consistent*.

(b) Assuming that Σ is consistent and that $\theta_1, \ldots, \theta_n, \ldots$ for $n \in \mathbb{N}$ lists all sentences of $\mathcal{L}$, prove that there is some maximal consistent Γ which contains Σ.

8. Let Γ be a maximal consistent set of sentences of $\mathcal{L}$. Assume that $\forall x\varphi(x) \in \Gamma$. Let $c \in \mathcal{L}$. Prove that $\varphi(c) \in \Gamma$.