MATH 445 – Final Exam – 15 December 2009

Work each numbered problem on a separate answer sheet. Show all your work for each problem clearly on the answer sheet for that problem. Write your name and the problem number on each answer sheet. Good luck.

Problems 1-3 are about sentential logic; the rest are about first order logic.

1. [25 pts] Determine which of the following are valid. Explain carefully — in particular, if the sentence is not valid, give an assignment falsifying it.

(a) $(A \to B) \to \big((C \to B) \to (A \to C)\big)$

(b) $\big((A \to B) \to (C \to B)\big) \to (A \to C)$

2. [20 pts] Find both the CNF and the DNF of the following sentence, and simplify if possible.

$\big(C \to (A \vee B)\big) \wedge \big(A \to (B \vee C)\big)$

3. [15 pts] Either prove (directly from the definitions) or disprove (with a counterexample) each of the following, where φ, ψ, and θ are sentences of sentential logic:

(a) If either $\varphi \models \theta$ or $\psi \models \theta$ then $(\varphi \wedge \psi) \models \theta$.

(b) If $(\varphi \to \psi)$ is satisfiable then ψ is satisfiable.

4. [25 pts] Let $\mathcal{L}$ be the language whose non-logical symbols are a binary function symbol F and a binary relation symbol R. Let $\mathcal{A}$ be the $\mathcal{L}$-structure $(\mathbb{Z}, +, <)$.

(a) Give a formula $\varphi(x)$ which defines the set of even integers in $\mathcal{A}$.

(b) Give a formula $\psi(x)$ which defines the set $\{1\}$ in $\mathcal{A}$.

(c) Give a sentence σ which is true on $\mathcal{A}$ but is false on the $\mathcal{L}$-structure $(\mathbb{Q}, +, <)$.

5. Working directly from the definitions, either prove or find a counterexample to each of the following. Explain carefully.

[15 pts] (a) $\models (\forall x P(x) \to \exists x Q(x)) \to \exists x (P(x) \to Q(x))$

[15 pts] (b) $\models \exists x (P(x) \to Q(x)) \to (\exists x P(x) \to \exists x Q(x))$

6. [15 pts] Either prove (diretly from the definitions) or disprove (with a counterexample) each of the following:

(a) $\exists x\varphi(x)$ is valid iff $\exists x\neg\varphi(x)$ is not valid.

(b) If $\models \forall x\big((\varphi(x) \vee \psi(x)) \rightarrow \theta(x)\big)$ then $\models \forall x(\varphi(x) \rightarrow \theta(x))$ and $\models \forall x(\psi(x) \rightarrow \theta(x))$.

7. [15 pts] (a) Define the function G by $G(k,l) =$ the largest n such that $n|k$ and $n|l$ provided $k \neq 0$ and $l \neq 0$, and $G(k,l) = 0$ otherwise. Prove that G is recursive.

[15 pts] (b) Show that the function F defined by $F(n) = \ulcorner\, \overline{n+2}\, \urcorner$ is recursive.

8. Let $F : \bar{\mathbb{N}} \rightarrow \bar{\mathbb{N}}$ be given.

(a) [5 pts] Define the course-of-values function $\overline{F}$.

(b) [15 pts] Prove that F is recursive iff $\overline{F}$ is recursive.

9. [5 pts] (a) State carefully the Theorem which gives the properties of the β-function needed in the proof of the Incompleteness Theorem.

[15 pts] (b) Let $\mathcal{A}$ be an $\mathcal{L}$-structure and Σ a set of sentences of $\mathcal{L}$ such that $\mathcal{A} \models \Sigma$. Assume that θ a sentence of $\mathcal{L}$ such that $\mathcal{A} \models \theta$ iff $\Sigma \not\models \theta$. Decide whether or not $\mathcal{A} \models \theta$ and explain carefully.

NOTE: Your solutions must include enough detail to justify your conclusions.