MATH 445 – Final Exam – 15 December 2010

Work each numbered problem on a separate answer sheet. Show all your work for each problem clearly on the answer sheet for that problem. Write your name and the problem number on each answer sheet. Good luck.

Problems 1-3 are about sentential logic; the rest are about first order logic.

1. [25 pts] Determine which of the following are valid. Explain carefully — in particular, if the sentence is not valid, give an assignment falsifying it.

(a) $\big((A \to (B \vee C))\big) \to \big(C \to \neg(B \to A)\big)$

(b) $\big(A \to (B \to C)\big) \vee \big(A \to (C \to B)\big)$

2. [20 pts] Find both the DNF and the CNF of the following sentence, and simplify if possible.

$A \to \big(B \to (A \wedge \neg C)\big)$

3. [15 pts] Either prove (directly from the definitions) or disprove (with a counterexample) each of the following, where φ, ψ, and θ are sentences of sentential logic:

(a) If $\varphi \models \theta$ and $\psi \models \theta$ then $(\varphi \vee \psi) \models \theta$.

(b) $\neg\varphi$ is satisfiable iff φ is not satisfiable.

4. [25 pts] Let $\mathcal{L}$ be the language whose only non-logical symbol is a binary function symbol F. Let $\mathcal{A}$ be the $\mathcal{L}$-structure $(\mathbb{N}, +)$.

(a) Give a formula $\varphi(x)$ of $\mathcal{L}$ which defines the set of odd integers in $\mathcal{A}$.

(b) Give a formula $\psi(x, y)$ of $\mathcal{L}$ which defines $<$ in $\mathcal{A}$, that is, such that $\mathcal{A} \models \psi(k, l)$ iff $k < l$.

(c) Give a sentence σ which is true on $\mathcal{A}$ but is false on the $\mathcal{L}$-structure $(\mathbb{Q}, +)$.

5. Working directly from the definitions, either prove or find a counterexample to each of the following. Explain carefully.

[15 pts] (a) $\models (\exists x P(x) \to \exists x Q(x)) \to \exists x (P(x) \to Q(x))$

[15 pts] (b) $\models (\forall x P(x) \to \forall x Q(x)) \to \forall x (P(x) \to Q(x))$

6. [15 pts] Either prove (diretly from the definitions) or disprove (with a counterexample) each of the following:

(a) $\forall x\varphi(x)$ is satisfiable iff $\forall x\neg\varphi(x)$ is not valid.

(b) If $\exists x(\varphi(x) \vee \psi(x))$ is valid then either $\exists x\varphi(x)$ is valid or $\exists x\psi(x)$ is valid (or both).

7. [15 pts] (a) Let $X \subseteq \mathbb{N}$ and define $F : \mathbb{N} \to \mathbb{N}$ by $F(n) =$ the number of elements of X which are $< n$. Assume that F is recursive. Prove that X is recursive.

[10 pts] (b) Recall that the recursive function $p : \mathbb{N} \to \mathbb{N}$ lists the primes in increasing order. Prove that the set P of primes is recursive. Explain carefully.

8. [15 pts] (a) Define $F : \mathbb{N} \to \mathbb{N}$ by $F(0) = F(1) = 1$, $F(2) = F(3) = 3$, $F(4) = F(5) = 5$, etc. Prove that F is recursive.

[15 pts] (b) Show that the function F defined by $F(n) = \ulcorner\, \overline{n+n}\, \urcorner$ is recursive. You may use the recursive function h_s which is such that $h_s(\ulcorner t \urcorner) = \ulcorner s(t) \urcorner$, but DO NOT USE the function *Num*.

9. [10 pts] (a) Let $F : \mathbb{N} \to \mathbb{N}$ be given. Recall that the course-of-values function $\overline{F}$ is defined so that $\overline{F}(n+1) = \langle F(0), \ldots, F(n)\rangle$. Assume that $\overline{F}$ is recursive. Prove that F is recursive.

[5 pts] (b) Carefully state the Incompleteness Theorem.

NOTE: Your solutions must include enough detail to justify your conclusions.