MATH 445 – Homework 4

(*due Monday 18 October 2010*)

1. [5 pts] Let $\mathcal{L}$ be the language whose non-logical symbols are a unary function symbol F and binary function symbols G and H. Let $\mathcal{A}$ be the $\mathcal{L}$-structure with universe $\mathbb{Z}$ which interprets F by $p(x)$ (where $p(k) = k - 1$), interprets G by $+$, and interprets H by $-$, and let $\mathcal{B}$ be the $\mathcal{L}$-structure with universe $\mathbb{Z}$ which interprets F by $s(x)$ (where $s(k) = k + 1$), interprets G by $\cdot$, and interprets H by $-$. Determine $t^{\mathcal{A}}$ and $t^{\mathcal{B}}$ where

$$t = H\big(G(F(x), F(y)), F(G(x, y))\big).$$

Simplify your answers.

2. Let φ be the formula

$$\big(\exists y R(x, y) \wedge \exists z R(y, z)\big) \to \forall x\big(R(x, z) \to \exists z R(z, x)\big).$$

[2 pts] (a) List all subformulas of φ.

[2 pts] (b) Determine which occurrences of variables in φ are free and which are bound.

3. [5 pts] Let $\mathcal{L}$ be the language whose only non-logical symbol is a binary relation symbol R. Let $\mathcal{A} = (\mathbb{N}, \leq)$. Determine the sets defined by the following formulas in $\mathcal{A}$. Explain and simplify your answers.

(a) $\varphi(x)$ is $\forall y(R(y, x) \to R(x, y))$

(b) $\psi(x)$ is $\forall y(R(x, y) \to R(y, x))$

4. [6 pts] Let $\mathcal{L}$ be the language whose only non-logical symbols are unary relation symbols E and P and the binary relation symbol R. Let $\mathcal{A}$ be the $\mathcal{L}$-structure with universe $\mathbb{N}$ which interprets E by the set of even natural numbers, P by the set of prime natural numbers, and R by $<$. Give sentences of $\mathcal{L}$ which "naturally" express the following facts about $\mathcal{A}$.

(a) There is a smallest prime number.

(b) There is no largest prime number.

(c) There is exactly one even prime number.

NOTE: Your solutions must include enough detail to justify your conclusions.