

MATH 446 – Exam II – 8 April 2011

Work each numbered problem on a separate answer sheet. Show all your work for each problem clearly on the answer sheet for that problem. Write your name and the problem number on each answer sheet. Good luck.

Important Note: No form of Choice is allowed on problems 1-3. The full AC can be used in problem 4.

1. [10 pts] (a) Explicitly define a linear order \leq^* on \mathbb{N} which is not a well order, but is such that \mathbb{N} has a smallest element under \leq^* . Justify your answer.

[15 pts] (b) Explicitly define a well order $\leq^\#$ on \mathbb{N} which has exactly two limit points. Justify your answer. Remember that 0_U is *not* a limit point of a well ordered set.

2. [15 pts] (a) Prove that every order-preserving injection π of a well ordered set U into itself is *expansive*, that is, satisfies $x \leq \pi(x)$ for all $x \in U$.

[10 pts] (b) Use the result in (a) to prove that no well ordered set is similar to a proper initial segment of itself.

3. [20 pts] Assume that for every set \mathcal{C} of non-empty pairwise disjoint sets there is a function $h : \mathcal{C} \rightarrow \bigcup \mathcal{C}$ such that $h(X) \in X$ for all $X \in \mathcal{C}$. Prove directly that the full Axiom of Choice holds.

4. (AC) Given indexed families $\{\kappa_i\}_{i \in I}$ and $\{\lambda_i\}_{i \in I}$ of cardinal numbers such that $\kappa_i < \lambda_i$ for all $i \in I$, establish the following. You may use the facts established in class about finite cardinal sums and products, but only the definition of the infinite sum.

[15 pts] (a) $\sum_{i \in I} \kappa_i \leq_c \sum_{i \in I} \lambda_i$.

[5 pts] Give an example to show we cannot conclude $<_c$ in (a). Justify your answer.

[10 pts] Assume that $\kappa_i =_c \kappa$ for all $i \in I$, where κ is some infinite cardinal, prove that $\sum_{i \in I} \kappa_i =_c \kappa \cdot |I|$.

NOTE: Your solutions must include enough detail to justify your conclusions.