MATH 446 – Final Exam – 17 May 2010

Work each numbered problem on a separate answer sheet. Show all your work for each problem clearly on the answer sheet for that problem. Write your name and the problem number on each answer sheet. Good luck.

Note: Choice is allowed only where stated. Replacement is not allowed in #3.

1. [20 pts] (a) Let $f : X \to Y$ be injective, and let $A, B \subseteq X$. Show that $f[A \smallsetminus B] = f[A] \smallsetminus f[B]$.

(b) Let $f : X \to Y$ and let $C \subseteq Y$. Show that $f\big[f^{-1}[C]\big] \subseteq C$, and determine under what conditions equality will hold.

2. [25 pts] (a) Prove (directly from the definitions) that $A <_c \mathcal{P}(A)$ holds for every set A.

(b) Assume that $A =_c A \times A$. Prove (directly from the definitions) that $\mathcal{P}(A) =_c (A \to A) =_c \big(A \to \mathcal{P}(A)\big)$.

3. [20 pts] Decide, using just Axioms I-VI, which of the following is a set and which a proper class. Explain carefully. You may use the fact that $\{A|\ A \text{ is a set}\}$ is a proper class.

(a) $\{a|\ \exists b\ (a, b) \in X\}$, where X is a set and (a, b) is the standard Kurotowski definition of ordered pair.

(b) $\{A|\ A \leq_c X\}$, where X is a non-empty set.

4. [25 pts] (a) Prove that every order-preserving map π of a well ordered set U into itself is *expansive*, that is, satisfies $x \leq \pi(x)$ for all $x \in U$.

(b) Use (a) to show that no well ordered set is similar to a proper initial segment of itself.

5. [20 pts] Recall that a set A is *Dedekind-infinite* provided that there is some injection of A into a proper subset of itself. Use Dependent Choice to show that every infinite set is Dedekind-infinite.

6. [20 pts] Recall that Zorn's Lemma states that if every chain in a p.o. set P has an upper bound, then P contains at least one maximal element. Prove (directly) that Zorn's Lemma implies the Axiom of Choice.

Hint: Given that $\forall x \in A\ \exists y \in B\ R(x, y)$ you want the elements of P to be certain functions on subsets of A into B.

7. [25 pts] (a) Prove that A is transitive iff $\mathcal{P}(A)$ is transitive for every pure set A.

(b) Prove (directly from the definitions) that if $\text{Rank}(A) = \alpha$ then $\text{Rank}(\mathcal{P}(A)) = \alpha + 1$.

8. [20 pts] Let $\alpha, \beta, \gamma \in ON$. Prove that if $0 < \alpha$ and $\alpha \cdot \beta = \alpha \cdot \gamma$ then $\beta = \gamma$.

Hint: First prove that if $\beta < \gamma$ and $0 < \alpha$ then $\alpha \cdot \beta < \alpha \cdot \gamma$.

9. [25 pts] Evaluate the following directly from the definitions of $+$ and $\cdot$ on the ordinals. Explain carefully.

(a) $\omega + \omega \cdot \omega$

(b) $\omega \cdot (\omega + 1)$

(c) $(\omega + 1) \cdot \omega$

NOTE: Your solutions must include enough detail to justify your conclusions.