MATH 446 – Homework 6

(*due MONDAY 12 April 2010*)

The following are to be solved using just Axioms I-VI, with no use of choice.

1. [8 pts] Prove that AC is equivalent to the following: for every $A \neq \emptyset$ and every $f : A \to B$ there is some $g : B \to A$ such that for all $a \in A$, $f(g(f(a))) = f(a)$.

2. [8 pts] Prove that AC is equivalent to the following: for every index set I and every indexed family of sets $\{A_i\}_{i \in I}$, if $A_i \neq \emptyset$ for all $i \in I$ then $\prod_{i \in I} A_i \neq \emptyset$.

3. [7 pts] We proved in class that DC implies the following: a p.o. set P is well-founded ("grounded") iff it has no infinite descending chains, that is there is no $f : \mathbb{N} \to P$ such that for every $n \in \mathbb{N}$ $f(n+1) <_P f(n)$. Prove the converse.

4. Define $\leq$ on $\mathbb{N} \to \mathbb{N}$ by $f \leq g$ iff $f = g$ or $\exists n \in \mathbb{N}$ such that $f(i) = g(i)$ for all $i < n$ but $f(n) < g(n)$.

[6 pts] (a) Prove that $\leq$ is a linear order of $\mathbb{N} \to \mathbb{N}$ but not a well order.

[1 pt] (b) Conclude that $\mathcal{P}(\mathbb{N})$ can be linearly ordered.

NOTE: Your solutions must include enough detail to justify your conclusions.