MATH 446 – Homework 6

(*due Monday 4 April 2011*)

The following are to be solved using just Axioms I-VI, with no use of choice.

1. [8 pts] Prove that AC is equivalent to the following: for every $A \neq \emptyset$ and every $f : A \to B$ there is some $g : B \to A$ such that for all $a \in A$, $f(g(f(a))) = f(a)$.

2. [8 pts] Prove that AC is equivalent to the following: for every index set I and every indexed family of sets $\{A_i\}_{i \in I}$, if $A_i \neq \emptyset$ for all $i \in I$ then $\prod_{i \in I} A_i \neq \emptyset$.

3. [7 pts] Recall that a p.o. set P is *well-founded*, or *grounded*, provided every non-empty $X \subseteq P$ contains a minimal element, that is, there is some $a \in X$ such that if $x \in X$ and $x \leq a$ then $x = a$. Assume, for every p.o. set P, that P is well-founded iff it has no infinite descending chains, that is, there is no $f : \mathbb{N} \to P$ such that for every $n \in \mathbb{N}$ $f(n+1) <_P f(n)$. Prove DC.

4. Define $\leq$ on $\mathbb{N} \to \mathbb{N}$ by $f \leq g$ iff $f = g$ or $\exists n \in \mathbb{N}$ such that $f(i) = g(i)$ for all $i < n$ but $f(n) < g(n)$.

[6 pts] (a) Prove that $\leq$ is a linear order of $\mathbb{N} \to \mathbb{N}$ but not a well order.

[1 pt] (b) Conclude that $\mathcal{P}(\mathbb{N})$ can be linearly ordered.

NOTE: Your solutions must include enough detail to justify your conclusions.