ELEMENTARY MATHEMATICAL LOGIC: INTRODUCTION AND OUTLINE

DAVID W. KUEKER

1. Introduction: What is Logic?

Mathematical Logic is, at least in its origins, the study of reasoning as used in mathematics. Mathematical reasoning is *deductive* — that is, it consists of drawing (correct) conclusions from given hypotheses. Thus the basic concept is that of a statement being a *logical consequence* of some other statements. In ordinary mathematical English the use of "therefore" customarily indicates that the following statement is a consequence of what comes before. For example:

Every integer is either odd or even. 7 is not even. Therefore 7 is odd.

To be a logical consequence the conclusion should not only be true (supposing the hypotheses to be true) but this should depend only on the "logical structure" of the statements — in this example, only on the meanings of "every", "or" and "not", but not on the meanings of "integer" "even", "odd", or "7". For example, consider the following:

Some integers are odd. Some integers are prime. Therefore some integers are both odd and prime.

Although the conclusion is true this is not a valid example of a logical consequence since the conclusion fails, although the hypotheses hold, if "prime" is replaced by "even".

To capture this aspect of logical consequence we will work in formal languages in which the "non-logical" symbols do not have a fixed meaning. A formal language determines a collection of *sentences* amd also a class of *interpretations* for the language. Each sentence of the formal langauge is either true or false in each interpretation, and we will define a sentence to be a *logical consequence* of a set of hypotheses if it is true in every interpretation which makes all of the hypotheses true.

Now, a *proof* (or *deduction*, the term we will use in dealing with formal languages) is an argument following certain specified rules. To be *sound*, the rules should guarantee that the results proved are in fact logical consequnces of the hypotheses assumed. The rules used must be explicitly and completely specified, so that it is possible to mechanically check whether a sequence of steps is really a proof.

The goal of mathematics is to show that certain statements (sentences) are true of some particular structure, or of each structure in some collection of structures. For example $|a + b| \leq |a| + |b|$ is true in the real numbers, but more generally true of any structure satisfying certain axioms.

The obvious question is: do proofs enable us to derive all sentences true of the structure, or collection of structures, in question? Of course this will depend on the formal language involved. Kurt Gödel gave two contrasting answers to this question, for first order languages. The first answer is the following:

Theorem 1. *(The Completeness Theorem) Let Σ be a set of first order sentences, and let θ be a first order sentence. Then θ is true in every model of Σ iff θ has a proof from Σ (in a proof system depending only on the language).*

But what if you want to know whether a sentence is true in some specific mathematical structure, such as the integers? His answer was the following surprising result:

Theorem 2. *(The Incompleteness Theorem) There is no axiomatic proof system strong enough to prove precisely the true sentences about arithmetic on the integers.*

Our goal in this course is to explain and prove these two theorems.

2. Outline

Although our main interest is in first order languages, we first study a simpler formal language, called *sentential logic*. We will define sentences, interpretations, logical consequence, and a proof system for sentential logic, and prove the Completeness Theorem (for sentential logic). This is the content of Chapter 1.

In Chapter 2 we introduce first order languages, and in Chapter 3 we prove Gödel's Completeness Theorem. In Chapter 4 we discuss computability and decidability, and in Chapter 5 we use this material to prove Gödel's Incompleteness Theorem.

3. References

What is Mathematical Logic?, by J. Crossley and others, is recommended reading. Chapter 1 is interesting for background, Chapter 2 covers the Completeness Theorem, although background in first order logic (for example from Chapter 2 of these Notes) is required, and Chapters 4 and 5 correspond to Chapters 4 and 5 of these notes, but here too we supply much necessary background.

There are (too) many texts on elementary mathematical logic. One standard reference is Enderton's *A Mathematical Introduction to Logic*, which covers all of the material in these notes more thoroughly, and in greater depth.