

CHAPTER 3. THE COMPLETENESS THEOREM

1. INTRODUCTION

In this Chapter we prove Gödel's Completeness Theorem for first order logic.

Theorem 1.1. (*Completeness Theorem*) *Let $\Sigma \subseteq Sn_{\mathcal{L}}$.*

- (a) *Σ is satisfiable iff Σ is consistent.*
- (b) *For any $\theta \in Sn_{\mathcal{L}}$, $\Sigma \vdash \theta$ iff $\Sigma \models \theta$.*

By Soundness (Theorem 9.2 in Chapter 2) we know the left to right implications of both (a) and (b). Recall also Lemma 7.2 and Theorem 10.4 from Chapter 2 which assert that $\Sigma \models \theta$ iff $(\Sigma \cup \{\neg\theta\})$ is not satisfiable and that $\Sigma \vdash \theta$ iff $(\Sigma \cup \{\neg\theta\})$ is not consistent. Therefore part (b) of Theorem 1.1 follows from part (a). So it suffices to establish the following result.

Theorem 1.2. (*Model Existence*) *Let $\Sigma \subseteq Sn_{\mathcal{L}}$ be consistent. Then Σ is satisfiable, (i.e., Σ has a model).*

As in sentential logic the argument for Theorem 1.2 will involve maximal consistent sets of sentences (see below), but we will have to expand the original consistent set by “adding witnesses”, a novel and important technique introduced by Leon Henkin in 1949.

The following definition is verbally the same as in sentential logic.

Definition 1.1. A set $\Gamma \subseteq Sn_{\mathcal{L}}$ is *maximal consistent* iff it is consistent and for every $\theta \in Sn_{\mathcal{L}}$ either $\theta \in \Gamma$ or $\neg\theta \in \Gamma$.

The lemma allowing us to extend consistent sets to maximal consistent sets is stated and proved exactly as in sentential logic.

Lemma 1.1. *Let $\Sigma \subseteq Sn_{\mathcal{L}}$ be consistent and let $\theta \in Sn_{\mathcal{L}}$. Then either $(\Sigma \cup \{\theta\})$ or $(\Sigma \cup \{\neg\theta\})$ is consistent.*

We next note that Finiteness is established exactly as for sentential logic.

Theorem 1.3. *Let $\Sigma \subseteq Sn_{\mathcal{L}}$.*

- (a) *For any $\theta \in Sn_{\mathcal{L}}$, $\Sigma \vdash \theta$ iff there is some finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vdash \theta$.*
- (b) *Σ is consistent iff every finite $\Sigma_0 \subseteq \Sigma$ is consistent.*

Finally the result on the existence of maximal consistent sets, due to Lindenbaum, is also stated and proved just as in sentential logic.

Theorem 1.4. *Let $\Sigma \subseteq Sn_{\mathcal{L}}$ be consistent. Then there is some maximal consistent $\Gamma \subseteq Sn_{\mathcal{L}}$ such that $\Sigma \subseteq \Gamma$.*

Note that Γ is not usually uniquely determined by Σ . For example, if both $\Sigma \cup \{\theta\}$ and $\Sigma \cup \{-\theta\}$ are consistent, then there will be maximal consistent sets Γ_1 containing $\Sigma \cup \{\theta\}$ and Γ_2 containing $\Sigma \cup \{-\theta\}$.

2. ADDING “WITNESSES”

In sentential logic we could define a (unique) truth assignment from a maximal consistent set Γ and then show that it satisfied Γ . In first order logic, however, a maximal consistent set does not determine any structure – to determine a structure we need to know what the universe of the structure is and how the non-logical symbols of the language are interpreted on this set.

Given a consistent set $\Sigma \subseteq Sn_{\mathcal{L}}$, we first expand the language \mathcal{L} to \mathcal{L}' by adding constant symbols c_i and find a maximal consistent set $\Gamma \subseteq Sn_{\mathcal{L}'}$ which also has the property that for every formula $\psi(x)$ of \mathcal{L}' , $\forall x\psi(x) \in \Gamma$ iff $\psi(c_i) \in \Gamma$ for every i . We will then be able to define an \mathcal{L}' -structure \mathcal{A}' such that every element of the universe is named by some constant symbol and which satisfies Γ .

To simplify the presentation we assume that the only non-logical symbol of the language \mathcal{L} is a binary relation symbol R . We start with a consistent set $\Sigma \subseteq Sn_{\mathcal{L}}$. We define the language $\mathcal{L}' = \mathcal{L} \cup \{c_i : i \in \mathbb{N}\}$. The consistency of Σ is not affected by this change since the set Σ says nothing about the added constants.

We next list all formulas of \mathcal{L}' with just one free variable as

$$\psi_0(x_0), \psi_1(x_1), \dots, \psi_n(x_n), \dots \text{ for all } n \in \mathbb{N}.$$

We define an increasing sequence Σ_n for $n \in \mathbb{N}$ of sets of sentences of \mathcal{L}' as follows:

$$\Sigma_0 = \Sigma;$$

to define Σ_1 we let c_{i_0} be the first constant not appearing in $\psi_0(x_0)$, let θ_0 be $(\exists x_0\psi_0(x_0) \rightarrow \psi_0(c_{i_0}))$, and let $\Sigma_1 = \Sigma_0 \cup \{\theta_0\}$; we claim that Σ_1 is consistent; otherwise, $\Sigma_0 \vdash \neg\theta_0$, hence $\Sigma_0 \vdash \exists x_0\psi_0(x_0)$ and $\Sigma_0 \vdash \neg\psi_0(c_{i_0})$; but then, by Generalization on Constants (Theorem 10.3 in Chapter 2), we would have $\Sigma_0 \vdash \forall x_0\neg\psi_0(x_0)$, and so Σ_0 is inconsistent (remembering that $\exists x$ means $\neg\forall x\neg$);

the general inductive step is similar; at the n th stage we have a consistent set Σ_n and we add some sentence θ_n of the form $\exists x_n\psi_n(x_n) \rightarrow \psi_n(c_{i_n})$, where c_{i_n} does not occur in Σ_n nor in $\psi_n(x_n)$, to obtain a consistent set Σ_{n+1} ; by Theorem 1.3 the union of all of these sets is a consistent set Σ' .

Now we apply Theorem 1.4 to obtain a maximal consistent $\Gamma \subseteq Sn_{\mathcal{L}'}$ such that $\Sigma' \subseteq \Gamma$.

We claim that Γ has the following properties:

- (i) for every $\theta \in Sn_{\mathcal{L}'}$, $\neg\theta \in \Gamma$ iff $\theta \notin \Gamma$,
- (ii) for every $\varphi, \theta \in Sn_{\mathcal{L}'}$, $(\varphi \rightarrow \theta) \in \Gamma$ iff either $\neg\varphi \in \Gamma$ or $\theta \in \Gamma$,
- (iii) for every $\psi(x) \in Fm_{\mathcal{L}'}$, $\forall x\psi(x) \in \Gamma$ iff $\psi(c_n) \in \Gamma$ for every $n \in \mathbb{N}$.

This claim is easily proved using the following Lemma.

Lemma 2.1. *Let Γ be maximal consistent. Then for any sentence θ , $\theta \in \Gamma$ iff $\Gamma \vdash \theta$.*

3. DEFINING A STRUCTURE FROM Γ

So, given Γ obtained as in the preceding section, we first define an \mathcal{L}' -structure \mathcal{A}' from Γ as follows:

the universe A' of the structure is \mathbb{N} ;

$c_n^{\mathcal{A}'} = n$ for every $n \in \mathbb{N}$;

$R^{\mathcal{A}'}(k, n)$ holds iff $R(c_k, c_n) \in \Gamma$.

We obtain the following result.

Theorem 3.1. *Let Γ and \mathcal{A}' be as above. Then for every sentence θ of \mathcal{L}' in which $=$ does not occur, $\mathcal{A}' \models \theta$ iff $\theta \in \Gamma$.*

Proof. (outline) We prove this by induction. The base case is $R(c_k, c_n)$, which holds due to the definition of $R^{\mathcal{A}'}$. The inductive steps for the connectives are clear from parts (i) and (ii) of the Claim at the end of the preceding section. The inductive step for \forall is clear from part (iii) of the Claim and the fact that $A' = \mathbb{N} = \{c_n^{\mathcal{A}'} : n \in \mathbb{N}\}$. \square

To define a structure which will also model the sentences in Γ which contain $=$ we need to allow for the possibility that $c_k = c_n \in \Gamma$ for some $k \neq n$. We chose the logical axioms for $=$ to guarantee that Γ has the following additional properties:

(iv) $c_k = c_k \in \Gamma$, if $c_k = c_n \in \Gamma$ then $c_n = c_k \in \Gamma$, and if $c_k = c_n, c_n = c_m \in \Gamma$ then $c_k = c_m \in \Gamma$;

(v) if $R(c_k, c_n), c_k = c_l$, and $c_n = c_m \in \Gamma$ then $R(c_l, c_m) \in \Gamma$.

We define a structure \mathcal{B}' as follows:

the universe B' of \mathcal{B}' is $\{k \in \mathbb{N} : c_k \neq c_i \in \Gamma \text{ for all } i < k\}$;

$c_n^{\mathcal{B}'}$ is the least k such that $(c_k = c_n) \in \Gamma$;

$R^{\mathcal{B}'}(k, l)$ holds iff $R(c_k, c_l) \in \Gamma$.

We then have the following, proved like Theorem 3.1 using the additional properties (iv) and (v) to check that equality statements in Γ are true in \mathcal{B}' .

Theorem 3.2. *For every $\theta \in Sn_{\mathcal{L}'}$. $\mathcal{B}' \models \theta$ iff $\theta \in \Gamma$.*

Since $\Sigma \subseteq \Gamma$ we have shown that the original Σ has a model, establishing Theorem 1.2.

Example. We illustrate the Henkin method with a simplified example. Let $\Sigma = \{\forall y \exists x R(y, x)\}$. Instead of listing all formulas $\psi(x)$ of \mathcal{L}' we consider only the formulas $\psi_n(x)$ defined as $R(c_n, x)$. Then θ_0 is $(\exists x R(c_0, x) \rightarrow R(c_0, c_1))$, and in general θ_n is $(\exists x R(c_n, x) \rightarrow R(c_n, c_{n+1}))$, so Σ' is $\{\forall y \exists x (R(y, x))\} \cup \{(\exists x R(c_n, x) \rightarrow R(c_n, c_{n+1})) : n \in \mathbb{N}\}$.

Now let $\Gamma \subseteq Sn_{\mathcal{L}'}$ be maximal consistent and contain Σ' . Then $\Gamma \vdash \exists x R(c_n, x)$ for every $n \in \mathbb{N}$ (since $(\forall y \exists x R(y, x) \rightarrow \exists x R(c_n, x))$ is an instance of Axiom 2). Therefore $\Gamma \vdash R(c_n, c_{n+1})$, and so $R(c_n, c_{n+1}) \in \Gamma$ by Lemma 2.1, for every $n \in \mathbb{N}$. Thus $\mathcal{A}' \models \forall y \exists x R(y, x)$, as desired, since $A' = \mathbb{N}$.

4. THE COMPACTNESS THEOREM

Just as in Sentential Logic, the Compactness Theorem is an immediate consequence of Finiteness and the Completeness Theorem.

Theorem 4.1. (Compactness Theorem) *Let Σ be a set of sentences of \mathcal{L} .*

- (a) *If $\Sigma \models \varphi$ then there is some finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \varphi$.*
- (b) *If every finite $\Sigma_0 \subseteq \Sigma$ has a model then Σ has a model.*

This result has many amazing applications and is of fundamental importance to the research area of Model Theory. We give two examples.

Theorem 4.2. *Let $\theta \in Sn_{\mathcal{L}}$. Assume that $\mathcal{A} \models \theta$ for every \mathcal{A} with an infinite universe A . Then there is some integer n such that $\mathcal{A} \models \theta$ for every \mathcal{A} whose universe A contains at least n elements.*

Proof. For every $k \in \mathbb{N}$ there is a sentence σ_k which holds of a structure iff the universe of the structure contains at least k elements. Let $\Sigma = \{\sigma_k : k \in \mathbb{N}\}$. Then $\mathcal{A} \models \Sigma$ iff the universe of \mathcal{A} is infinite. Therefore the hypothesis of the Theorem implies that $\Sigma \models \theta$. By Compactness there is some finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \theta$. There must be a largest integer n such that $\sigma_n \in \Sigma_0$, and this n is then as desired. \square

Theorem 4.3. *Let \mathcal{L} be the language whose only non-logical symbol is a binary relation symbol $<$. Let \mathcal{A} be the \mathcal{L} -structure $(\mathbb{N}, <)$. Then there is some $\mathcal{B} = (B, <^{\mathcal{B}})$ such that $\mathcal{B} \models \theta$ for every θ true on \mathcal{A} , but \mathcal{B} contains “infinite” elements, that is elements b with infinitely many elements in B preceding it in the order $<^{\mathcal{B}}$.*

Proof. For every $n \in \mathbb{N}$ there is a formula $\varphi_n(x)$ of \mathcal{L} which holds of an element iff there are at least n elements preceding it. Let c be a constant symbol and let $\mathcal{L}' = \mathcal{L} \cup \{c\}$. We define the set Σ of \mathcal{L}' -sentences as

$$\{\theta \in Sn_{\mathcal{L}} : \mathcal{A} \models \theta\} \cup \{\varphi_n(c) : n \in \mathbb{N}\}.$$

If $\mathcal{B}' \models \Sigma$ then $\mathcal{B} = (B', <^{B'})$ is as desired, since $c^{B'}$ is an “infinite” element of B' . Every finite $\Sigma_0 \subseteq \Sigma$ has a model $\mathcal{A}' = (\mathbb{N}, <, n_0)$ where n_0 is a sufficiently large element of \mathbb{N} , so Σ has a model by Compactness. \square

The reader should try to think of what such a \mathcal{B} would look like.