APPENDIX: AXIOMS FOR ARITHMETIC ON THE NATURAL NUMBERS

1. Introduction

In this appendix we give an axiomatic system for arithmetic on the *natural numbers*, that is, the set $\mathbb{N}$ of non-negative integers. The arithmetic functions we consider are s (successor), $+$, and $\cdot$. We also include the order relation $<$ and the constant 0 (of course, $=$ is always included). The non-logical symbols of the first order language $\mathcal{L}_{\mathbb{N}}$ for arithmetic on the natural numbers are s, $+$, $\cdot$, $<$, and $\bar{0}$. We will also write $x + y$, $x < y$ (instead of $+(x, y)$, $<(x, y)$), etc. to make it easier to read formulas. We use $\mathcal{N}$ for the $\mathcal{L}_{\mathbb{N}}$-structure $(\mathbb{N}, s, +, \cdot, 0, <)$.

2. The Axioms

We introduce two groups of axioms. The first contains the "algebraic" or "arithmetic" properties of the basic functions and relations.

Q1. $\forall x(\neg s(x) = \bar{0})$
Q2. $\forall x \forall y(s(x) = s(y) \to x = y)$
Q3. $\forall x(x + \bar{0} = x)$
Q4. $\forall x \forall y(x + s(y) = s(x + y))$
Q5. $\forall x(x \cdot \bar{0} = \bar{0})$
Q6. $\forall x \forall y(x \cdot s(y) = x \cdot y + x)$
Q7. $\forall x(\neg x < \bar{0})$
Q8. $\forall x \forall y(x < s(y) \leftrightarrow x < y \vee x = y)$
Q9. $\forall x \forall y(x < y \vee x = y \vee y < x)$

This collection of nine axioms will be referred to as **Q**.

The second group of axioms is infinite and consists of all instances of *induction*.

IS consists of all sentences of the following form where $\varphi(x, z_1, \ldots, z_n)$ is a formula of $\mathcal{L}_{\mathbb{N}}$:

$$\forall z_1, \ldots \forall z_n \big(\varphi(\bar{0}, \vec{z}) \wedge \forall x(\varphi(x, \vec{z}) \to \varphi(s(x), \vec{z})) \to \forall x \varphi(x, \vec{z})\big)$$

Q together with **IS** is called *Peano Arithmetic*, abbreviated **PA**.

Clearly, $\mathcal{N} \models \mathbf{PA}$. Furthermore, for any sentence σ that you are likely to encounter in number theory, if $\mathcal{N} \models \sigma$ then $\mathbf{PA} \models \sigma$. However, Kurt Gödel in 1931 published his amazing Incompleteness Theorem which implies that there are sentences σ such that $\mathcal{N} \models \sigma$ but $\mathbf{PA} \not\models \sigma$.

3. Arithmetic in **PA**.

In this section we give a few examples of how arithmetic on the natural numbers can be developed in **PA**. Note, of course, that since $\mathcal{N} \models \mathbf{PA}$ we know that $\mathcal{N} \models \sigma$ whenever $\mathbf{PA} \models \sigma$. We define the terms $\bar{1}$, $\bar{2}$, etc to be $s(\bar{0})$, $s(s(\bar{0}))$, etc., so every natural number n is named in $\mathcal{N}$ by the variable-free term $\bar{n}$ of $\mathcal{L}_{\mathbf{N}}$. We leave it to the reader to supply a proof of the following.

Lemma 3.1. *It $k = l$ then $\mathbf{Q} \models \bar{k} = \bar{l}$. If $k \neq l$ then $\mathbf{Q} \models \neg(\bar{k} = \bar{l})$.*

The next result lists some provable properties of addition.

Lemma 3.2. (a) *If $k + l = n$ then $\mathbf{Q} \models \bar{k} + \bar{l} = \bar{n}$. If $k + l \neq n$ then $\mathbf{Q} \models \neg(\bar{k} + \bar{l} = \bar{n})$.*
(b) $\mathbf{PA} \models \forall x \forall y (x + y = y + x)$
(c) $\mathbf{PA} \models \forall x \forall y \forall z \big((x + y) + z = x + (y + z)\big)$

Proof. (a) We show that "for every k and n if $k+l = n$ then $\mathbf{Q} \models \bar{k}+\bar{l} = \bar{n}$" holds for all l, by induction on l. The base case is $l = 0$: if $k + 0 = n$ then $k = n$, and $\mathbf{Q} \models \bar{k} + \bar{0} = \bar{k}$ as desired since this is axiom **Q3.** For the inductive step we assume for some fixed l that "for every k and n if $k+l = n$ then $\mathbf{Q} \models \bar{k} + \bar{l} = \bar{n}$" and we show the same property for $l + 1$. Suppose that $k + l + 1 = n$. Then $n = m + 1$ where $k + l = m$. By the inductive hypothesis $\mathbf{Q} \models \bar{k} + \bar{l} = \bar{m}$, and thus by axiom **Q4**, $\mathbf{Q} \models \bar{k} + s(\bar{l}) = s(\bar{m})$, which is as desired since $\overline{l + 1}$ is $s(\bar{l})$ and $\bar{n}$ is $s(\bar{m})$. The second assertion is easier (why?).

(b) (outline) We first prove two preliminary facts:
(i) $\mathbf{PA} \models \bar{0} + x = x$, and
(ii) $\mathbf{PA} \models \big((s(x) + y = s(x + y)\big)$.
Both of these use induction (in (ii) on y) and axioms **Q3** and **Q4**. Using these we then show that $\mathbf{PA} \models (x + y = y + x)$ which we do by induction on y, with x as a parameter. By Generalization this yields the conclusion. Details are left to the reader.

(c) (outline) By Generalization it suffices to prove that $\mathbf{PA} \models \big((x + y) + z = x + (y + z)\big)$, which we do by induction on z. Details are left to the reader. □

We have a similar result about multiplication.

Lemma 3.3. (a) *If $k \cdot l = n$ then $\mathbf{Q} \models \bar{k} \cdot \bar{l} = \bar{n}$. If $k \cdot l \neq n$ then $\mathbf{Q} \models \neg(\bar{k} \cdot \bar{l} = \bar{n})$.*
(b) $\mathbf{PA} \models \forall x \forall y (x \cdot y = y \cdot x)$.
(c) $\mathbf{PA} \models \forall x \forall y \forall z \big((x \cdot y) \cdot z = x \cdot (y \cdot z)\big)$.

As another example we state some results about $<$.

Lemma 3.4. (a) $\mathbf{PA} \models \forall x \neg(x < x)$
(b) $\mathbf{PA} \models \forall x \forall y \forall z (x < y \wedge y < z \to x < z)$

(c) $\mathbf{PA} \models \forall x \forall y (x < y \to \neg y < x)$
(d) *For every* $n > 0$, $\mathbf{Q} \models \forall x \big(x < \bar{n} \leftrightarrow (x = \bar{0} \vee x = \bar{1} \vee \cdots \vee x = \overline{n-1})\big)$

Corollary 3.1. *If* $k < l$ *then* $\mathbf{Q} \models (\bar{k} < \bar{l})$. *If* $k \not< l$ *then* $\mathbf{Q} \models \neg(\bar{k} < \bar{l})$.

4. Defining Relations and Functions in PA

An n-ary relation R on $\mathbb{N}$ is *definable in* $\mathcal{N}$ provided there is some $\varphi(x_1, \ldots, x_n) \in Fm_{\mathcal{L}_\mathbb{N}}$ such that for all $k_1, \ldots, k_n \in \mathbb{N}$, $R(k_1, \ldots, k_n)$ holds iff $\mathcal{N} \models \varphi(\bar{k}_1, \ldots, \bar{k}_n)$. We say that an n-ary function F on $\mathbb{N}$ is *definable in* $\mathcal{N}$ provided there is a formula $\varphi(x_1, \ldots, x_n, y)$ such that for all $k_1, \ldots, k_n$ and $l \in \mathbb{N}$, $F(k_1, \ldots, k_n) = l$ iff $\mathcal{N} \models \varphi(\bar{k}_1, \ldots, \bar{k}_n, \bar{l})$.

For some purposes we require a stronger property which states that whether the sentence $\varphi(\bar{k}_1, \ldots, \bar{k}_n)$ or its negation is true in $\mathcal{N}$ depends only on the axioms in **PA**. We call this stronger property *definability in* **PA**.

Definition 4.1. An n-ary relation R on $\mathbb{N}$ is *definable in* **PA** *by the formula* $\varphi(x_1, \ldots, x_n)$ iff for every $k_1, \ldots, k_n \in \mathbb{N}$ the following are equivalent:
(i) $R(k_1, \ldots, k_n)$ holds,
(ii) $\mathcal{N} \models \varphi(\bar{k}_1, \ldots, \bar{k}_n)$,
(iii) $\mathbf{PA} \models \varphi(\bar{k}_1, \ldots, \bar{k}_n)$.

Note that (*iii*) always implies (*ii*) since $\mathcal{N} \models \mathbf{PA}$. Thus to show that φ defines R in **PA** it suffices to show that φ defines R in $\mathcal{N}$ and that (*i*) implies (*iii*) holds.

Lemma 3.1 implies that the relation $k = l$ is definable in **PA** by the formula $x = y$. Corollary 3.1 implies that the relation $k < l$ is definable in **PA** by the formula $x < y$.

As another example we show that the divisibility relation $k|l$ is definable in **PA**.

Lemma 4.1. *There is a formula* $\delta(x, y)$ *defining the divisibility relation* $k|l$ *in* **PA**.

Proof. We define $\delta(x, y)$ as $\exists z (y = x \cdot z)$. Suppose that $k|l$. Then there is some n such that $l = k \cdot n$. It follows by Lemma 3.3(a) that $\mathbf{PA} \models (\bar{l} = \bar{k} \cdot \bar{n})$ and thus $\mathbf{PA} \models \delta(\bar{k}, \bar{l})$.

If $\mathcal{N} \models \delta(\bar{k}, \bar{l})$ then there is some $n \in \mathbb{N}$ such that $\mathcal{N} \models (\bar{l} = \bar{k} \cdot \bar{n})$ and so $l = k \cdot n$ and $k|n$. □

Using Lemma 4.1 we easily conclude that there is a formula $\pi(x)$ defining the set of primes in **PA**.

We say that the n-ary function F is definable in **PA** provided the $(n+1)$-ary relation $F(x_1, \ldots, x_n) = y$ is definable in **PA**. For example, Lemmas 3.2(a) and 3.3(a) imply that the functions $+$ and $\cdot$ are definable in **PA** by the formulas $x + y = z$ and $x \cdot y = z$ respectively.