AMSC/CMSC 460: Final Exam

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May 17, 2018

Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 2 hours.
- Good luck!

Additional instructions:

- The exam has 2 parts: part A and part B. Each part has 4 problems.
- You should solve only 3 out of the 4 problems in each part.
- No extra credit will be given for solving more than 3 problems in each part.
- If you solve more than 3 problems, you should clearly indicate which problems you would like to be graded - otherwise, the first 3 problems in each part will be graded.

Part A: Choose 3 problems out of problems 1-4 (Each problem = 10 points)

1. Find the most accurate approximation to $f'(x)$ using $f(x-\frac{h}{2}), f(x), f(x+h)$. What is the order of accuracy of this approximation?

2. Find a quadrature of the form
$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = A_0 f(x_0) + A_1 f(x_1),$$
that is exact for all polynomials of degree ≤ 3.

3. (a) Write the Lagrange form of the linear interpolation polynomial that interpolates $f(x)$ at $x = -1, 1$.
   (b) Use the interpolant you obtained in part (a) to find a weighted quadrature of the form
$$\int_{-2}^{2} x f(x) dx = A_0 f(-1) + A_1 f(1).$$

4. Find a linear polynomial, $P_1^*(x)$, that minimizes
$$\int_{-\infty}^{\infty} e^{-x^2} (x^3 - Q_1(x))^2 dx,$$
among all polynomials $Q_1(x)$ of degree ≤ 1.

Part B: Choose 3 problems out of problems 5-8 (Each problem = 10 points)

5. Find values for a, b, c, d such that the following function, $s(x)$, is a cubic spline on $[0, 2]$ that satisfies $s'(2) = 0$,
$$s(x) = \begin{cases} x^3 - ax^2 + b, & 0 \leq x \leq 1, \\ cx^3 + dx^2, & 1 \leq x \leq 2. \end{cases}$$

6. Use the Gram-Schmidt process to find <u>orthonormal</u> polynomials of degrees 0 and 1 with respect to the inner product
$$\langle f, g \rangle_w = \int_0^{\infty} f(x) g(x) e^{-2x} dx.$$

7. Explain what the floating point representation of $\frac{1}{10}$ looks like on a 32-bit machine.

8. Find a Cholesky decomposition of
$$A = \begin{pmatrix} 16 & 12 & 4 \\ 12 & 13 & 3 \\ 4 & 3 & 17 \end{pmatrix}.$$

- Chebyshev polynomials

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) = 0, \ \forall n \geq 1.$$

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = 0, \quad m \neq n.$$

$$\int_{-1}^{1} \frac{(T_n(x))^2}{\sqrt{1-x^2}} dx = \begin{cases} \pi, & n = 0, \\ \frac{\pi}{2}, & n = 1, 2, \ldots \end{cases}$$

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi.$$

- Hermite polynomials

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \ \forall n \geq 1$$

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) = \delta_{nm} 2^n n! \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} x^m e^{-x^2} dx = \Gamma\left(\frac{m+1}{2}\right), \quad \text{for even } m$$

$$\Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(3/2) = \tfrac{1}{2}\sqrt{\pi}, \quad \Gamma(5/2) = \tfrac{3}{4}\sqrt{\pi}.$$

- Other formulas

$$\int x e^{ax} dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right).$$

$$\int x^2 e^{ax} dx = \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right).$$