AMSC/CMSC 460: Midterm 1 Solutions

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Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 75 minutes
- Good luck!

Problems: (Each problem = 10 points)

1. (a) Write the number 35.35 in base 2. (Compute the first 10 digits after the binary point).

   Solution: 100011.0101100110

   (b) Explain how 35.35 can be represented as a floating point number on a 64-bit computer.

   Solution:
   Using binary scientific notation: $(35.35)_{10} = (1.00011010110011... \times 2^5)_2$ This means that a floating point representation will have the following components: (i) one sign bit (in this case 0). (ii) a designated number of bits for the exponent (in this case 5, which will be stored as 0...0101) (iii) the remaning bits for the mantissa. Skipping the leading 1, we are left with 00011010100110011... It is a good idea to specify the number of bits in each part.

   (c) Explain how 35.35 can be stored with a fixed point representation on a 64-bit computer. What are the advantages of a floating point representation over a fixed point representation?

   Solution: With fixed point - divided the word in two: for example, use 32 bits for the number before the dot, and 32 bits for the number after the dot. In this case, the representation will be: 0...0100011 for the first 32 bits, and 01011001100... for the last 32 bits.

   As of the advantages of the floating point representation, the important issues to mention are the range of numbers that can be represented and the precision.

   (d) Explain two approaches for representing the (negative) number -35 on a computer with a 64-bit word.

   Solution: Option 1 - Use a sign bit: 10...0100011. Option 2 - With 2's complement: 1...1011101.

2. Consider the following matrix A, and its inverse A^{-1}:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 3 \\ 0 & 2 & -1 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -1/6 & 1/6 & 2/3 \\ -1/3 & 1/3 & 1/3 \end{pmatrix}$$

(a) Compute the condition number of A in the infinity norm.

Solution: Since $\|A\|_\infty = 4$ and $\|A^{-1}\|_\infty = 2$,

$$\kappa_\infty(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty = 8.$$

(b) Find an LU decomposition of A where L is a unit lower triangular matrix.

Solution:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 4 \\ 0 & 0 & 3 \end{pmatrix}.$$

(c) Use the LU decomposition that you found, to solve $Ax = b$ with $b = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}$.

Solution: We want to solve $Ax = b$, but now that we decomposed A as $A = LU$, we can write $LUx = b$. We set $Ly = b$ and solve for y: $y = \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix}$.

Finally, we solve $Ux = y$ to conclude: $x = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$.

3. Let $f(x) = e^{-x} - x^2$.

(a) Prove that there exists at least one point $x^* \in [0, 10]$ for which $f(x^*) = 0$.

Solution: We note that $f(x)$ is a continuous function. We also note that $f(0) > 0$ while $f(10) < 0$. By the intermediate value theorem, this applies that there exists $x^* \in [0, 10]$ such that $f(x^*) = 0$.

(b) Starting from $x_0 = 0$, use Newton's method to compute two approximations x_1 and x_2 for a root of $f(x)$.

Solution: We use the general form for Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This leads to $x_1 = 1$ and $x_2 = 1 - \dfrac{e^{-1} - 1}{-e^{-1} - 2}$.

(c) Starting from $x_0 = 0$ and $x_1 = 1$, compute one iteration of the secant method for the given function $f(x)$.

Solution: Replacing the derivative in Newton's method for the slope of the tangent connecting x_{n-1} and x_n we have

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}.$$

In this case, starting from $x_0 = 0$ and $x_1 = 1$, we get $x_2 = 1 - \frac{e^{-1} - 1}{e^{-1} - 2}$.

4. Let $f(x) = \cos(\pi x)$.

   Let $x_0 = -1, x_1 = 0, x_2 = 1$, and let $y_j = f(x_j)$ for $j = 0, 1, 2$.

(a) Write Newton's form for the interpolation polynomial, $P_2(x)$, that interpolates the data at the three given points.

Solution: First we evaluate $f(x)$ at the given points: $f(x_0) = -1, f(x_1) = 1, f(x_2) = -1$.
The corresponding divided differences are: $f[x_0] = -1, f[x_0, x_1] = 2$ and $f[x_0, x_1, x_2] = -2$. Hence

$$P_2(x) = -1 + 2(x + 1) - 2(x + 1)x = 1 - 2x^2.$$

(b) Write Lagrange's form for the interpolation polynomial, $P_2(x)$, that interpolates the data at the three given points.

Solution:

$$P_2(x) = -1\frac{x(x-1)}{2} + 1\frac{(x+1)(x-1)}{-1} - 1\frac{(x+1)x}{2} = 1 - 2x^2.$$

(c) Verify that the answers to parts (a) and (b) are identical. Explain the advantages of Newton's form over Lagrange's form.

Solution: Simplifying the expressions in parts (a) and (b) shows that the polynomials are identical, as should be. With regards to the advantages of Newton's form over Lagrange's form, there are several points that can be made. The most important one is the structure that allows to relatively easily add a point. Newton's method is also easier to program and there are several things that can be said about that.

(d) Write the Lagrange form of the polynomial $P_3(x)$, that interpolated $f(x) = \sin(\pi x)$ at the four points: $x_0 = -1, x_1 = 0, x_2 = 1, x_3 = 1/2$.

Solution: Since $f(x_0) = f(x_1) = f(x_2) = 0$ and $f(x_3) = 1$, the Lagrange form of the interpolating polynomial $P_3(x)$ has only one term:

$$P_3(x) = \frac{(x+1)x(x-1)}{\frac{3}{2}\frac{1}{2}\left(-\frac{1}{2}\right)} = \frac{8}{3}x(1-x^2).$$