

AMSC/CMSC 460: Midterm 2

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**Read carefully the following instructions:**

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 75 minutes
- Good luck!

**Problems: (Each problem = 10 points)**

- (a) Explain the advantages of interpolating at Chebyshev points.  
(b) Compute the unique interpolating polynomial of degree  $\leq 2$  that interpolates data sampled from  $f(x) = x^2$  at an appropriate number of Chebyshev points on the interval  $[-1, 1]$ .  
(c) Repeat part (b) with  $f(x) = x^4$ .

Note: Chebyshev polynomials are given by

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) = 0, \forall n \geq 1.$$

- Find a spline of degree 2,  $S(x)$ , on the interval  $[0, 2]$ , for which  $S(0) = 0$ ,  $S(1) = 2$ ,  $S(2) = 0$ , and  $S'(0) = 0$ . Use the points 0, 1, 2 as the knots.
- Use the Gram-Schmidt process to find orthogonal polynomials of degrees 0, 1, 2, on the interval  $[0, 1]$ , with respect to the weight  $w(x) = 1 + x$ .

Note: you do not need to normalize the polynomials. For the quadratic polynomial,  $P_2(x)$ , write the coefficients but do not explicitly calculate the integrals.

- Let  $f(x) = x^2$ . Find the quadratic polynomial  $Q_2^*(x)$  that minimizes

$$\int_{-\infty}^{\infty} e^{-x^2} (f(x) - Q_2(x))^2 dx,$$

among all quadratic polynomials  $Q_2(x)$ .

Note: You may use:

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \forall n \geq 1$$

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x) dx = \delta_{nm}2^n n! \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} x^m e^{-x^2} dx = \Gamma\left(\frac{m+1}{2}\right), \quad \text{for even } m$$

$$\Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(3/2) = \frac{1}{2}\sqrt{\pi}, \quad \Gamma(5/2) = \frac{3}{4}\sqrt{\pi}.$$