AMSC 466: Final Exam
Prof. Doron Levy
December 17, 2010

Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 2 hours
- Good luck!

Part I

<u>Instructions:</u> For all problems in Part I, write the answer in your exam book and a very short explanation of your solution. A correct solution with no explanation will not be accepted as a correct answer.

1. (2 points) Let $f(x) = (\sin x)^2$ in the interval $I = [-1, 1]$. We are interested in finding a root of $f(x)$ in the interval I and for that we use Newton's method with an arbitrary starting point $x_0 \in I$. Denote by x_n the approximate root at stage n. Then

   (a) $\lim_{n\to\infty} x_n = \pi$.

   (b) The series $\{x_n\}$ does not converge.

   (c) $\lim_{n\to\infty} x_n = 0$.

   (d) There is not enough information to decide.

2. (2 points) We approximate the integral of $f(x)$ in $I = [a, b]$ using the following algorithm:

   - Split the interval I into two equal intervals
   $$I_1 = [a, (a+b)/2], \quad I_2 = [(a+b)/2, b].$$
   - Approximate $\int_{I_1} f(x)dx$ using the composite midpoint rule (with subintervals of length h).
   - Approximate $\int_{I_2} f(x)dx$ using the composite Simpson's rule (with subintervals of length h).
   - Finally, use the previous two approximations to write
   $$\int_I f(x)dx = \int_{I_1} f(x)dx + \int_{I_2} f(x)dx.$$

   Assume that $f(x)$ is differentiable 10 times. What is the order of the method?

   (a) 4.

   (b) 3.

   (c) 2.

   (d) Not enough information to decide.

3. (2 points) How many different LU decompositions exist for the matrix $A = \begin{pmatrix} \mathrm{a} & 0 \\ 0 & \mathrm{a} \end{pmatrix}$, with constant a? (L is lower triangular and U is upper triangular. Nothing else is known about L and U.).

   (a) 1.

   (b) ∞.

   (c) 0.

   (d) Depends on the value of a.

4. (2 points) Given a function $f(x)$ in $[a, b]$, we sample it in three points, $a = x_0 < x_1 < x_2 = b$, and construct the following three approximations:

$$\begin{aligned} A^1(x) &= \text{The quadratic interpolation polynomial through } x_j,\ j = 0, 1, 2. \\ A^2(x) &= \text{The spline of degree 1 with knots } x_j,\ j = 0, 1, 2. \\ A^3(x) &= \text{The linear least-squares approximation of } f(x) \text{ in } [a, b]. \end{aligned}$$

   For each approximation i ($i = 1, 2, 3$) we denote the error at any point x by $E^i(x) = f(x) - A^i(x)$. Then

   (a) $|E^1(x_j)| \leq |E^3(x_j)|$ for $j = 0, 1, 2$.

   (b) $E^1(x_j) = E^2(x_j)$ for $j = 0, 1, 2$.

   (c) (a) and (b) are correct.

   (d) There is not enough information to decide

Part II

1. (4 points) Derive a quadrature of the form

$$\int_{-1}^{1} f(x)dx \approx Af\left(-\frac{1}{4}\right) + Bf(0) + Cf\left(\frac{1}{4}\right),$$

   that has the highest possible accuracy.

2. (4 points) Compute the higest order approximation to the first derivative of $f(x)$ at a point a that is based on the values of $f(x)$ at the three points: $f(a-2h)$, $f(a-h)$, $f(a)$, with a constant $h > 0$. What is the order of your approximation?

3. (4 points) Compute the Cholesky decomposition of

$$A = \begin{pmatrix} 15 & -18 & 15 \\ -18 & 24 & -18 \\ 15 & -18 & 18 \end{pmatrix}$$

4. (6 points) Let $f(x) = \sin x$ in $[0, \pi]$.

   (a) (2 points) Find the first two orthonormal polynomials with respect to the weight function $w(x) \equiv 1$ on $[0, \pi]$.

   (b) (4 points) Let $r_1^*(x)$ denote the linear polynomial that minimizes

$$\int_0^{\pi} [\sin x - r_1(x)]^2 dx,$$

   among all linear polynomials $r_1(x) \in \Pi_1$. Explain why the function $r_1^*(x)$ must be a constant function, and compute it.