AMSC/CMSC 460: Final Exam

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Read carefully the following instructions:

- Write your name & student ID on the exam book and sign it.
- You may not use any books, notes, or calculators.
- Solve all problems. Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- Exam time: 120 minutes
- Good luck!

Problems:

1. **(10 points)** Consider the following 2 values of a function $f(x)$: $f(1)$ and $f(1+h)$.

   (a) Find the Lagrange form of the polynomial that interpolates the given values.

   (b) Use the results of part (a) to derive a quadrature for $\int_{1-h}^{1+h} f(x)dx$.

2. **(10 points)** Let $w(x) = 1$, $\forall x \in [0,1]$.

   (a) Use the Gram-Schmidt process to find the first two orthonormal polynomials, $P_0(x)$ and $P_1(x)$ (of degrees 0 and 1, respectively), with respect to the inner product

$$\langle f(x), g(x) \rangle_w = \int_0^1 f(x)g(x)w(x)dx.$$

(b) Use the least squares theory to find the linear polynomial $Q_1(x)$, that minimizes

$$\int_0^1 (e^x - Q_1(x))^2 dx.$$

(c) Use the results of part (a) to find the most accurate quadrature of the form

$$\int_0^1 f(x)dx \approx A_0 f(x_0).$$

3. **(10 points)**

(a) Consider the ODE, $y'(t) = f(t, y(t))$ for $a \leq y \leq b$, subject to the initial value $y(a) = y_0$. Explain how to obtain the modified Euler method for approximating solutions of this initial-value problem, by using the midpoint quadrature rule on the integral form of the ODE.

(b) Assume an approximation

$$f''(x) = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} + c_1 h^2 + c_2 h^4 + O(h^6).$$

Use Richardson's extrapolation to find an $O(h^4)$ approximation to $f''(x)$. Write explicitly the resulting approximation.

4. **(10 points)**

(a) Use divided differences with repetitions to find a polynomial of a minimal degree that interpolates $f(0) = f'(0) = f(1) = f'(1) = 1$.

(b) Find a Cholesky decomposition for

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 6 \end{pmatrix}$$