AMSC 466: Final Exam

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Read carefully the following instructions:

- Write your name & student ID on the exam book, copy the honor pledge and sign it.
- You may not use any books, notes, or calculators.
- Answer all problems after carefully reading them. Start every problem on a new page.
- Show all your work and explain everything you write.
- The maximum grade is 100.
- Exam time: 2 hours
- Good luck!

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Problems:

1. (20 points). Find a formula of the form

$$\int_{-1}^{1} f(x)dx \approx A_0 f(0) + A_1 f(1) + A_2 f(2),$$

that is exact for all functions of the form $f(x) = ax + bx^3 + c\cos\frac{\pi x}{2}$.

2. (20 points). Consider the following 3 values of $f(x)$: $f(x-h), f(x)$, and $f(x+2h)$.

   (a) Using these values, find the best approximation of $f'(x)$.
   What is the order of this approximation?

   (b) Using these values, find the best approximation of $f''(x)$.
   What is the order of this approximation?

   (c) Using these values, find any approximation of $f'(x) + f''(x)$.
   What is the order of this approximation?

3. (20 points). Consider the following matrix:

$$A = \begin{pmatrix} 4 & 10 & 12 \\ 10 & 50 & 40 \\ 12 & 40 & 100 \end{pmatrix}$$

Find a Cholesky decomposition for A.

4. (20 points) Let $T_n(x)$ be the Chebyshev polynomial of degree n. Let x_0, x_1 be the roots of $T_2(x)$.

   (a) Find x_0, x_1.

   (b) Write the Lagrange form for interpolating a function $f(x)$ at x_0, x_1. Find a bound on the interpolation error in the interval $[-1, 1]$.

   (c) Find a Gaussian quadrature of the form

   $$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx \approx Af(x_0) + Bf(x_1),$$

   where x_0 and x_1 are the roots of $T_2(x)$. If $f(x)$ is a polynomial, what is the degree of $f(x)$ for which this Gaussian quadrature is exact?

In solving this problem you may use the recursion relation for Chebyshev polynomials:

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0.$$

You may also use the following formula

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c.$$

Note that for $n \neq m$, $\displaystyle\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = 0.$

5. (a) (10 points). Let $w(x) = e^{-x}$. Find the first two orthogonal polynomials with respect to the inner product

   $$\langle f(x), g(x) \rangle_w = \int_0^{\infty} f(x)g(x)w(x)dx.$$

   (Do not normalize the polynomials).

   (b) (10 points). Find the polynomial of degree ≤ 1, $p_1(x)$, that minimizes

   $$\int_0^{\infty} e^{-x}(e^{-x} - p_1(x))^2 dx.$$

In solving both parts of this problem you may use the following formula:

$$\int x^n e^{ax} dx = \frac{e^{ax}}{a}\left(x^n - \frac{nx^{n-1}}{a} + \frac{n(n-1)x^{n-2}}{a^2} - \cdots \frac{(-1)^n n!}{a^n}\right) + c, \quad n = \text{positive integer}$$