Midterm 3 - Solutions

(1) $f(x) = xe^{-x^2}$.

(a) Domain : $-\infty < x < \infty$

(b) $x = 0 \Rightarrow f(0) = 0$.
$f(0) = 0 \Rightarrow x = 0$ $\Longrightarrow$ $(0,0)$ is the only x & y intercept.

(c) No vertical asymptotes.

(d) $\lim\limits_{x \to \pm\infty} f(x) = \lim\limits_{x \to \pm\infty} \frac{x}{e^{x^2}} = 0$. $\Longrightarrow$ $y = 0$ is a horizontal asymptote at $x \to \pm\infty$.

(e) $f'(x) = (1 - 2x^2) e^{-x^2}$
$\Longrightarrow f'(x) = 0$ when $1 - 2x^2 = 0$
$\Rightarrow 2x^2 = 1 \Rightarrow \underline{\underline{x = \pm\sqrt{\frac{1}{2}}}}$.

At $\sqrt{\frac{1}{2}}$ $f(\sqrt{\frac{1}{2}}) = \sqrt{\frac{1}{2}}\, e^{-(\sqrt{\frac{1}{2}})^2} = \frac{1}{\sqrt{2}} e^{-\frac{1}{2}}$.

At $\sqrt{-\frac{1}{2}}$ $f(-\sqrt{\frac{1}{2}}) = -\sqrt{\frac{1}{2}}\, e^{\frac{1}{2}} = -\frac{1}{\sqrt{2}} e^{-1/2}$.

(f) f': $-\downarrow$ on $x < -\frac{1}{\sqrt{2}}$, $+\uparrow$ on $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$, $-\downarrow$ on $x > \frac{1}{\sqrt{2}}$

$f(x)$ increasing: $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

$f(x)$ decreasing: $x < -\frac{1}{\sqrt{2}}$ or $x > \frac{1}{\sqrt{2}}$.

(g) $f''(x) = 2x(2x^2-3)e^{-x^2}$

$f''(x) = 0 \Rightarrow x = 0$ or $2x^2-3=0$

$\Rightarrow x = \pm\sqrt{\frac{3}{2}}$.

f'': $-$ on $x<-\sqrt{\frac{3}{2}}$, $+$ on $-\sqrt{\frac{3}{2}}<x<0$, $-$ on $0<x<\sqrt{\frac{3}{2}}$, $+$ on $x>\sqrt{\frac{3}{2}}$

$f(x)$ Concave up: $-\sqrt{\frac{3}{2}} < x < 0$ or $x > \sqrt{\frac{3}{2}}$.

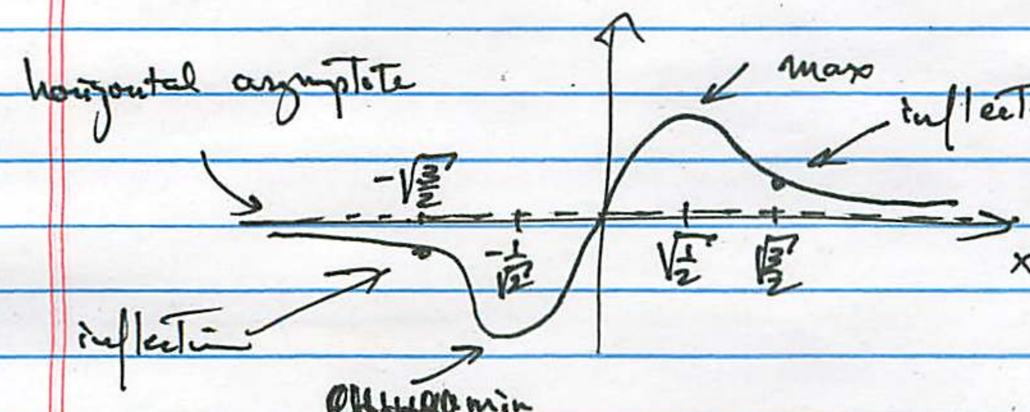
$f(x)$ concave down $x < -\sqrt{\frac{3}{2}}$ or $0 < x < \sqrt{\frac{3}{2}}$.

(h) $x = -\sqrt{\frac{3}{2}}$ $\quad f(-\sqrt{\frac{3}{2}}) = -\sqrt{\frac{3}{2}}e^{-\frac{3}{2}}$

$x = 0$ $\quad f(0) = 0$

$x = \sqrt{\frac{3}{2}}$ $\quad f(\sqrt{\frac{3}{2}}) = \sqrt{\frac{3}{2}}e^{\frac{3}{2}}$

(i)



(2) (a) $y^2x^3 + e^{xy} = 3$

$$\Rightarrow 2y\frac{dy}{dx}x^3 + y^2\cdot 3x^2 + ye^{xy} + x\frac{dy}{dx}e^{xy} = 0.$$

$$\frac{dy}{dx}\left(2yx^3 + xe^{xy}\right) = -3y^2x^2 - ye^{xy}$$

$$\boxed{\frac{dy}{dx} = -\frac{3y^2x^2 + ye^{xy}}{2yx^3 + xe^{xy}}}.$$

(b) $3x^2 + \ln y = x^3y^2 + 2$

$$\Rightarrow 6x + \frac{1}{y}\frac{dy}{dx} = 3x^2y^2 + x^3 2y\frac{dy}{dx}$$

At (1,1): $\quad 6 + \frac{dy}{dx} = 3 + 2\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx}(1,1) = 6-3 = 3.$$

i.e. the slope of the tangent line at (1,1) is 3.

$\Rightarrow$ The tangent line is:

$$y - 1 = 3(x-1) = 3x - 3$$

$$\boxed{y = 3x - 2}$$

(3) (a) $f(x) = \frac{x}{x^2+4}$

$$f'(x) = \frac{1\cdot(x^2+4) - x\cdot 2x}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2} = 0$$

$\Rightarrow \quad x = \pm 2.$

$f(2) = \frac{2}{2^2+4} = \frac{2}{8} = \frac{1}{4}$ ← local max

$f(-2) = \frac{-2}{(-2)^2+4} = -\frac{1}{4}$ ← local min.

(b) The point $x=-2$ is outside the given interval,

$\Rightarrow$ The only relative extreme we should look at is $x=2$.

To this we add the boundary points 0,4:

x	$f(x)$	
0	0	$\Rightarrow$ absolute min
2	$\frac{1}{4}$	$\Rightarrow$ absolute max.
4	$\frac{4}{16+4} = \frac{1}{5}$	

(4) (a) $T(x) = \dfrac{2+x}{2+x^2}$

$$\frac{dT}{dt} = \frac{dT}{dx} \cdot \frac{dx}{dt}$$

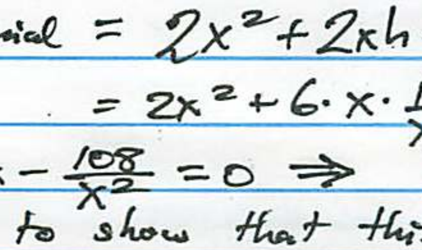
It is given that $\frac{dx}{dt} = 4$ and that $x = 3$.

$$\frac{dT}{dx} = \frac{2+x^2 - 2x(2+x)}{(2+x^2)^2} = \frac{2+x^2 - 4x - 2x^2}{(2+x^2)^2}$$

$$= \frac{-x^2 - 4x + 2}{(2+x^2)^2}$$

$$\frac{dT}{dx}(x=3) = \frac{-9-12+2}{(2+9)^2} = \frac{-19}{121}$$

$$\frac{dT}{dt}(x=3) = \left.\frac{dT}{dx}\right|_{x=3} \cdot \frac{dx}{dt} = \frac{-19}{121} \cdot 4 = \underline{\underline{-\frac{76}{121}}}$$

(b) 

Volume $= 2x^2h = 36$
$\Rightarrow h = \dfrac{18}{x^2}$.

$A(x) = \text{Material} = 2x^2 + 2xh + 4xh = 2x^2 + 6xh$
$= 2x^2 + 6 \cdot x \cdot \dfrac{18}{x^2} = 2x^2 + \dfrac{108}{x}$

$A'(x) = 4x - \dfrac{108}{x^2} = 0 \Rightarrow x^3 = \dfrac{108}{4} = 27 \Rightarrow x = 3$

You have to show that this is a minimum (eg. $A''(3) > 0$).

$\Rightarrow$ The dimensions are : $x = 3$ ft, $2x = 6$ ft, $h = \frac{18}{9} = 2$ ft