Midterm #2 – Solutions

1. (a) $f(x) = 6x^2 - 4$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} =$$

$$= \lim_{h\to 0} \frac{6(x+h)^2 - 4 - (6x^2-4)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{6x^2} + 12xh + 6h^2 \cancel{-4} \cancel{-6x^2} \cancel{+4}}{h}$$

$$= \lim_{h\to 0} \left[12x + 6h\right] = 12x.$$

(b) $f(x) = \frac{2}{x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h}$$

$$= \lim_{h\to 0} \frac{2x - 2(x+h)}{h\cdot(x+h)x} = \lim_{h\to 0} \frac{-2}{(x+h)x}$$

$$= -\frac{2}{x^2}$$

2. (a) $y(x)=\dfrac{e^x-e^{-x}}{x}$

$$y'(x)=\frac{(e^x-e^{-x})'x-x'(e^x-e^{-x})}{x^2}$$

$$=\frac{(e^x+e^{-x})x-(e^x-e^{-x})}{x^2}$$

(b) $y(t)=t\,3^{\sqrt{t}}$ =

$$y'(t)=t'\cdot 3^{\sqrt{t}}+t\left(3^{\sqrt{t}}\right)'$$

$$=3^{\sqrt{t}}+t\cdot 3^{\sqrt{t}}\ln 3\,\tfrac{1}{2}t^{-1/2}.$$

(c) $y(x)=\left(\ln|x+1|\right)^4$

$$y'(x)=4\left(\ln|x+1|\right)^3\frac{1}{x+1}.$$

(d) $y=\sqrt{\dfrac{\sin x}{\sin 3x}}=\left(\dfrac{\sin x}{\sin 3x}\right)^{1/2}$

$$y'(x)=\frac{1}{2}\left(\frac{\sin x}{\sin 3x}\right)^{-1/2}\frac{(\sin x)'\sin 3x-\sin x\cdot(\sin 3x)'}{(\sin 3x)^2}.$$

$$=\frac{1}{2}\left(\frac{\sin x}{\sin 3x}\right)^{-1/2}\frac{\cos x\cdot\sin 3x-\sin x\cdot 3\cdot\cos 3x}{(\sin 3x)^2}.$$

3. (a) $f(x) = -\frac{3}{e^{x+1}} = -\frac{3}{e} e^{-x}$. $x=0$.

(i) point At $x=0$ $f(0) = -\frac{3}{e}$.

(ii) slope. $f'(x) = \frac{3}{e} e^{-x}$

At $x=0$ $f'(0) = \frac{3}{e}$.

(iii) Equation: $y - f(0) = f'(0)(x-0)$

$\Rightarrow y + \frac{3}{e} = \frac{3}{e} \cdot x \Rightarrow y = \frac{3}{e}(x-1)$.

(b) $f(x) = \frac{x^3 - 1}{2\ln x}$, $x = e$.

(i) point At $x=e$ $f(e) = \frac{e^3-1}{2\ln e} = \frac{e^3-1}{2}$.

(ii) slope

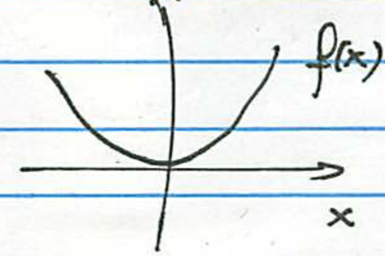
$$f'(x) = \frac{1}{2} \frac{3x^2 \cdot \ln x - (x^3-1)\cdot\frac{1}{x}}{(\ln x)^2}$$

$$f'(e) = \frac{1}{2}\left[3e^2 - (e^3-1)\cdot\frac{1}{e}\right] = \frac{1}{2}\left[2e^2 + \frac{1}{e}\right] = e^2 + \frac{1}{2e}.$$

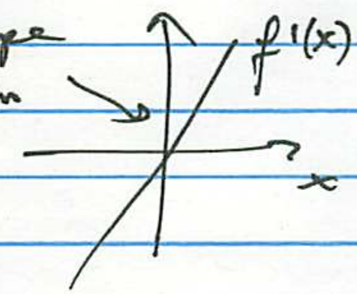
(iii) equation: $y - f(e) = f'(e)(x-e)$

$\Rightarrow y - \frac{e^3-1}{2} = \left(e^2 + \frac{1}{2e}\right)(x-e)$.

4. (a)



zero slope at origin



(b) $f(x) = \cos(2x)$

Average rate of change over $[0, \frac{\pi}{2}]$:

$$\frac{\cos\left(2\cdot\frac{\pi}{2}\right) - \cos(2\cdot 0)}{\frac{\pi}{2} - 0} = \frac{\cos\pi - \cos 0}{\frac{\pi}{2}} =$$

$$= \frac{-1-1}{\frac{\pi}{2}} = \frac{-2}{\frac{\pi}{2}} = -\frac{4}{\pi}.$$

Instantaneous rate of change at $x=0$

$$f'(0) = -2\sin(2x)\Big|_{x=0} = 0.$$

(c) $f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}.$

Hence, $f(x)$ is differentiable at any $x \neq 0$.
At $x > 0$ $f'(x) = 1$. At $x < 0$ $f'(x) = -1$.
At $x = 0$, the slopes are different from both sides
$\Longrightarrow$ no derivative.