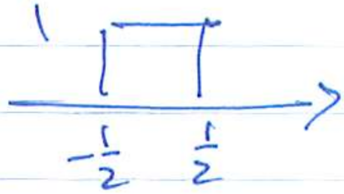
Chp. 3 The Fourier TRANSFORM

$$F(s) = \int_{x=-\infty}^{\infty} f(x)\, e^{-2\pi i s x}\, dx \qquad \text{FT.}$$

$$f(x) = \int_{s=-\infty}^{\infty} F(s)\, e^{2\pi i s x}\, ds \qquad \text{IFT}$$

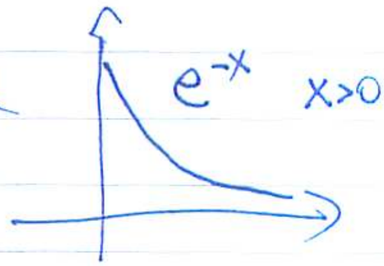
Examples: (1) The box function

$$\Pi(x) = \begin{cases} 1 & -\frac{1}{2} < x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



$$F(s) = \operatorname{sinc}(s) = \frac{\sin(\pi s)}{\pi s}.$$

(2) Truncated exponential



$e^{-x} \cdot h(x)$

$$h(x) = \begin{cases} 1 & x>0 \\ 0 & x<0. \end{cases} \qquad \text{heaviside function}$$

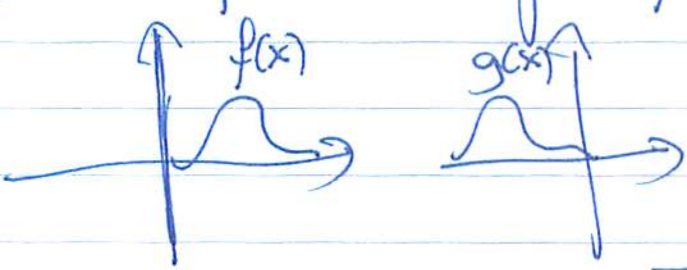
$$\boxed{F(s) = \frac{1}{1+2\pi i s}}$$

(3) Gaussian $f(x) = e^{-\pi x^2} \Rightarrow F(s) = e^{-\pi s^2}.$

Rules:

1. Linearity

2. Reflection: $g(x) = f(-x)$, $G(s) = F(-s)$.



3. Conjugation: $g(x) = \overline{f(x)} \Rightarrow G(s) = \overline{F(-s)}$

4. Translation: $g(x) = f(x - x_0) \Rightarrow G(s) = e^{-2\pi i s x_0} F(s)$

5. Modulation: $g(x) = e^{2\pi i s_0 x} f(x) \Rightarrow G(s) = F(s - s_0)$.

6. Dilation: $g(x) = f(ax) \Rightarrow G(s) = \frac{1}{|a|} F\left(\frac{s}{a}\right)$

7. Inversion: $g(x) = F(x) \Rightarrow \cancel{G(s) = f(-s)}$

$$\Rightarrow G(s) = f(-s).$$

8. Derivative: $g(x) = f'(x)$

$$\Rightarrow G(s) = 2\pi i s F(s).$$

$\Uparrow$!

(9) Convolution: $g(x) = (f_1 * f_2)(x)$

$$G(s) = F_1(s) \cdot F_2(s).$$

(10) <u>Multiplication:</u> $g(x) = f_1(x) \cdot f_2(x)$

$$\Rightarrow G(s) = (F_1 * F_2)(s).$$

$$G(s) \overset{\text{def}}{=} \int_{x=-\infty}^{\infty} f_1(x) f_2(x) e^{-2\pi i s x} dx =$$

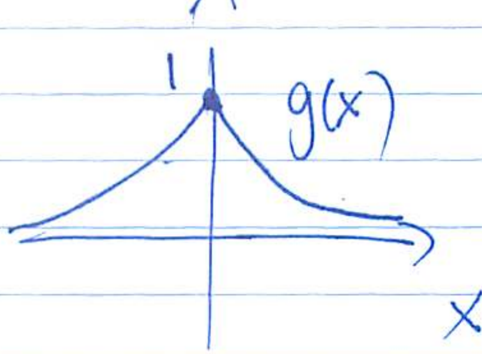
$$= \int_{x=-\infty}^{\infty} \underbrace{\left(\int_{\sigma=-\infty}^{\infty} F_1(\sigma) e^{2\pi i \sigma x} d\sigma \right)}_{f_1(x)} f_2(x) e^{-2\pi i s x} dx =$$

$$= \int_{\sigma=-\infty}^{\infty} F_1(\sigma) \underbrace{\left[\int_{x=-\infty}^{\infty} f_2(x) e^{-2\pi i (s-\sigma) x} dx \right]}_{F_2(s-\sigma)} d\sigma \overset{\text{def } *}{=} (F_1 * F_2)(s).$$

Examples

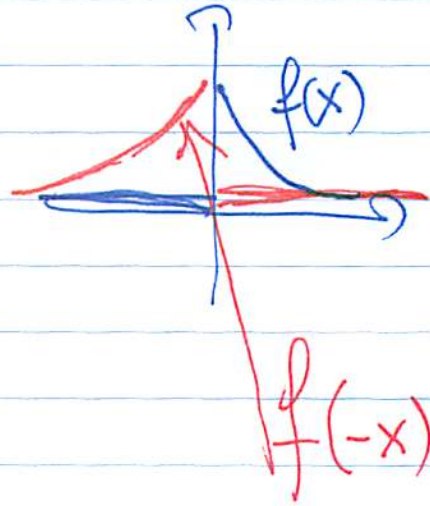
① $g(x) = e^{-|x|}$

$G(s) = ?$



We know: $f(x) = e^{-x} h(x)$

$$F(s) = \frac{1}{1+2\pi i s}.$$



$$g(x) = f(x) + f(-x)$$

$$G(s) = F(s) + F(-s)$$

$$= \frac{1}{1+2\pi i s} + \frac{1}{1-2\pi i s} =$$

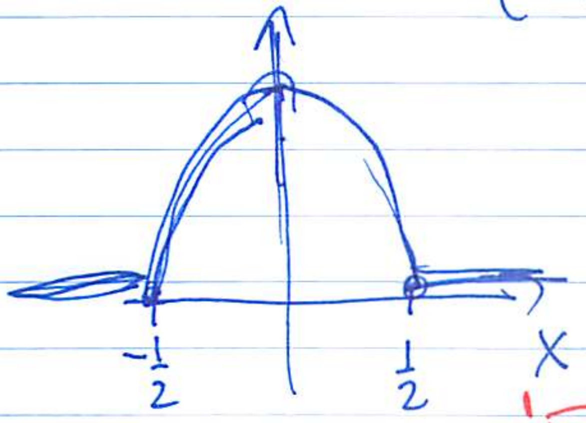
$$= \frac{(1-\cancel{2\pi i s}) + (1+\cancel{2\pi i s})}{(1+2\pi i s)(1-2\pi i s)} = \frac{2}{1-(2\pi i s)^2} =$$

$$= \frac{2}{1+4\pi^2 s^2}$$

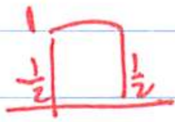
Alt.: $G(s) = \int_{-\infty}^{\infty} e^{-|x|} e^{-2\pi i s x}\, dx$

(2).

$$g(x)=\begin{cases}\cos\pi x & -\frac{1}{2}<x<\frac{1}{2}\\ 0 & \text{otherwise.}\end{cases}$$



$$G(s)=\int_{x=-\frac{1}{2}}^{\frac{1}{2}}\cos\pi x\, e^{-2\pi i s x}\,dx$$



$$g(x)=\cos\pi x\cdot\Pi(x)=\underbrace{\frac{e^{i\pi x}+e^{-i\pi x}}{2}}_{\cos\pi x}\cdot\Pi(x)$$

$$\left[\left.\begin{aligned} e^{i\pi x}&=\cos\pi x+\cancel{i\sin\pi x}\\ e^{-i\pi x}&=\cos(-\pi x)+i\sin(-\pi x)\\ e^{-i\pi x}&=\cos(\pi x)-\cancel{i\sin(\pi x)}\end{aligned}\right\}\Rightarrow\cos\pi x=\frac{e^{i\pi x}+e^{-i\pi x}}{2}\right]$$

$$g(x)=\frac{1}{2}e^{i\pi x}\,\Pi(x)+\frac{1}{2}e^{-i\pi x}\,\Pi(x)$$

const.

sinc(s)

Translation.

$$g(x) = e^{2\pi i s_0 x} f(x) \Rightarrow G(s) = F(s - s_0)$$

$$\underbrace{e^{i\pi x}\Pi(x)}_{\operatorname{sinc}(s-\frac{1}{2}).} = e^{2\pi i \underset{s_0}{\frac{1}{2}} x}\Pi(x)$$

$$g(x) = \frac{1}{2}e^{i\pi x}\Pi(x) + \frac{1}{2}e^{-i\pi x}\Pi(x)$$

$$e^{2\pi i \cdot \underset{= s_0}{\frac{1}{-2}} \cdot x}$$

$$\boxed{G(s) = \frac{1}{2}\operatorname{sinc}(s-\tfrac{1}{2}) + \frac{1}{2}\operatorname{sinc}(s+\tfrac{1}{2})}$$

$$s - s_0 = s - (-\tfrac{1}{2}) = s + \tfrac{1}{2}.$$

$$\left(\operatorname{sinc}(s) = \frac{\sin \pi s}{\pi s}\right) \rightarrow F(\Pi(x)).$$

$$\frac{1}{2}\frac{\sin\left[\pi\left(s-\frac{1}{2}\right)\right]}{\pi\left(s-\frac{1}{2}\right)} + \dots$$

Normal Dist.

$$g(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

translation

scaling

μ = mean.
σ = std. dev.

$$\left(\int_{-\infty}^{\infty} g(x)\,dx = 1\right).$$

const.

We know: $f(x) = e^{-\pi x^2} \Rightarrow F(s) = e^{-\pi s^2}$.

Approach I

Step I: Ignore μ.

$$h(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}.$$

$$\boxed{\begin{aligned} g(x) &= f(ax) \\ G(s) &= \frac{1}{|a|} F\left(\frac{s}{a}\right) \end{aligned}}$$

$$h(x) = \frac{1}{\sqrt{2\pi}\,\sigma} f\left(\frac{x}{\sqrt{2\pi}\,\sigma}\right)$$

$$e^{-\pi\left[\frac{x}{\sqrt{2\pi}\,\sigma}\right]^2} = e^{\frac{-\pi x^2}{2\pi\sigma^2}} = e^{-\frac{x^2}{2\sigma^2}}. \checkmark$$

$$a = \frac{1}{\sqrt{2\pi}\,\sigma}$$

$$H(s) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \frac{1}{\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)} \cdot F\left(\frac{s}{a}\right)$$

$$= \cancel{\frac{1}{\sqrt{2\pi}\,\sigma}\sqrt{2\pi}\,\sigma} \cdot F\left(\frac{s}{a}\right) = F\left(\frac{s}{\frac{1}{\sqrt{2\pi}\,\sigma}}\right) = F\left(\sqrt{2\pi}\,\sigma s\right)$$

$$= e^{-\pi\left[\sqrt{2\pi}\,\sigma s\right]^2} = e^{-\pi\cdot 2\cdot \pi\cdot \sigma^2 s^2}$$

$$= e^{-2\pi^2\sigma^2 s^2}$$

$$g(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad h(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

$$G(s) = H(s)\cdot e^{-2\pi i s\mu}$$

$$\boxed{G(s) = e^{-2\pi i s\mu}\, e^{-2\pi^2\sigma^2 s^2}}$$

Approach II: WRONG! 1st μ. 2nd scaling.

$$e^{-\pi x^2} \rightarrow e^{-\pi(x-\mu)^2} \rightarrow e^{-\pi\left(\frac{x}{2\sigma^2}-\mu\right)^2}$$

$f(x)=e^{-|x|}$ $\qquad$ $F(s)=\dfrac{2}{1+4\pi^2 s^2}$.

$g(x)=e^{-2|x-3|}$. $\qquad$ $\cancel{e^{-|2x-}}$

$h(x)=\underline{e^{-2|x|}} = e^{-|2x|} = f(2x)$

$H(s)=\dfrac{1}{|a|}F\left(\dfrac{s}{a}\right) = \dfrac{1}{2}F\left(\dfrac{s}{2}\right) = \dfrac{1}{\cancel{2}}\cdot\dfrac{\cancel{2}}{1+4\pi^2\left(\frac{s}{2}\right)^2}$

$a=2$ $\qquad$ $= \underline{\underline{\dfrac{1}{1+\pi^2 s^2}}}$.

$g(x)=h(x-3)$

$G(s)=H(s)\cdot e^{-2\pi i s\cdot 3}$

$= \underline{\underline{e^{-6\pi i s}\cdot\dfrac{1}{1+\pi^2 s^2}}}$.

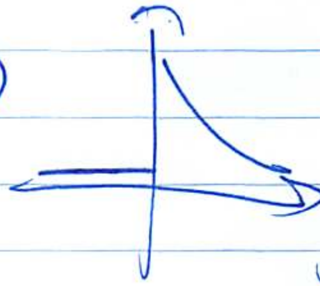
Example: (Inversion $g(x) = F(x)$, $G(s) = f(-s)$)

1) $g(x) = \operatorname{sinc} x = \dfrac{\sin \pi x}{\pi x}$

$G(s) = \Pi(-s) = \Pi(s)$ Box function.

2) $f(x) = e^{-x} h(x)$

$F(s) = \dfrac{1}{1+2\pi i s}$.



} Known.

$g(x) = \dfrac{1}{1+2\pi i x} \Rightarrow G(s) = f(-s)$

$= e^{+s} h(-s)$.

3) $$\boxed{\begin{aligned} g(x) &= \frac{2}{1+4\pi^2 x^2} \\ G(s) &= f(-s) = e^{-|-s|} \\ &= e^{-|s|}. \end{aligned}}$$

HW: $\dfrac{1}{1+x^2}$ FT = ?