Scaling: $\quad f(x) \longrightarrow F(s).$

$$g(x) = x \cdot f(x) \longrightarrow G(s) = ?$$

$$G(s) = \int_{-\infty}^{\infty} g(x) e^{-2\pi i s x} dx = \int_{-\infty}^{\infty} x f(x) e^{-2\pi i s x} dx$$

$$\left[-2\pi i \underbrace{x e^{-2\pi i s x}} = \frac{d}{ds}\left(e^{-2\pi i s x}\right) \right]$$

$$\frac{1}{-2\pi i} \frac{d}{ds}\Bigg(\underbrace{\int_{-\infty}^{\infty} f(x) e^{-2\pi i s x} dx}_{F(s)} \Bigg) = -\frac{1}{2\pi i} F'(s).$$

Example: $\quad f(x) = e^{-\pi x^2} \Rightarrow F(s) = e^{-\pi s^2}.$

$$g(x) = x e^{-\pi x^2} \Rightarrow G(s) = ?$$

$$G(s) = -\frac{1}{2\pi i} F'(s) = -\frac{1}{2\pi i}\left(e^{-\pi s^2}\right)' = -\frac{1}{\cancel{2\pi} i} \cdot \left(-\cancel{\pi} \cdot \cancel{2} s\right) e^{-\pi s^2}$$

$$= \frac{s}{i} e^{-\pi s^2} = -s i e^{-\pi s^2}.$$

Examples (§3.3)

I. Evaluation of integrals

(1). Evaluate : $I(\alpha) = \underbrace{\int_{-\infty}^{\infty} e^{-\alpha x^2} dx}$, $\alpha > 0$.

$$f(x) = e^{-\alpha x^2}$$

$$I(\alpha) = \int_{-\infty}^{\infty} f(x) \overbrace{e^{-2\pi i \cdot 0 \cdot x}}^{1} dx \quad = F(0)$$

$$\left[F(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x} dx \right]$$

$$f(x) = e^{-\alpha x^2} = e^{-\pi \left(\frac{\sqrt{\alpha}\, x}{\sqrt{\pi}}\right)^2}$$

$$g(x) = e^{-\pi x^2} \Rightarrow G(s) = e^{-\pi s^2}$$

$$f(x) = g\left(\frac{\sqrt{\alpha}\, x}{\sqrt{\pi}}\right) \Rightarrow F(s) = \frac{1}{\sqrt{\frac{\alpha}{\pi}}} e^{-\pi \left(\sqrt{\frac{\pi}{\alpha}}\, s\right)^2}$$

$$= \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\pi^2 s^2}{\alpha}}.$$

$$\Rightarrow I(\alpha) = F(0) = \sqrt{\frac{\pi}{\alpha}}$$

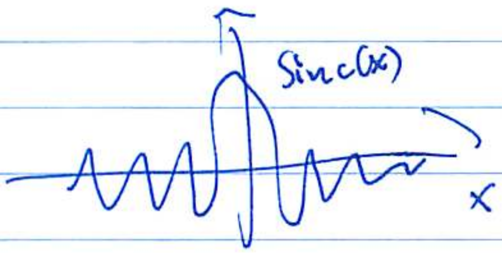
s

② $$I(\alpha) = \int_{-\infty}^{\infty} \frac{\sin(\pi x)\cos(2\pi\alpha x)}{\pi x}\,dx, \quad -\infty < \alpha < \infty.$$

$f(x) = \mathrm{sinc}(x)$

$$f(x) = \mathrm{sinc}(x) \Rightarrow F(s) = \Pi(s)$$

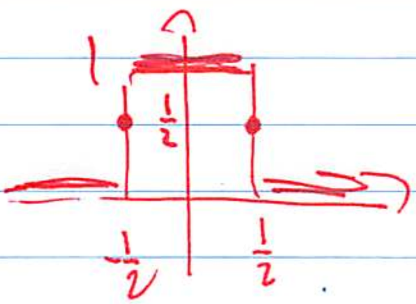
$$I(\alpha) = \int_{-\infty}^{\infty} f(x)\cos(2\pi\alpha x)\,dx = \boxed{\underbrace{\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \alpha x}\,dx}_{\text{even}}} = F(\alpha)$$



$$\cos(-2\pi\alpha x) + i\sin(-2\pi\alpha x)$$

even odd

$$I(\alpha) = F(\alpha) = \Pi(\alpha) = \begin{cases} 1 & |\alpha| < \frac{1}{2} \\ \frac{1}{2} & |\alpha| = \frac{1}{2} \\ 0 & |\alpha| > \frac{1}{2} \end{cases}$$ ← LATER



(3). $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$

$f(x) = \dfrac{1}{1+x^2}$ $\qquad \displaystyle\int_{-\infty}^{\infty} (f(x))^2\, dx$

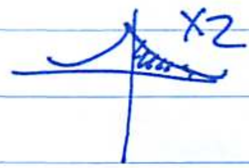
Plancherel: $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$

easier (in this case).

$F(s) = \pi e^{-2\pi|s|}$.

$g(x) = e^{-|x|}$ $\qquad x \to s.$

$G(s) = \dfrac{2}{1+4\pi^2 s^2}$



$$\int_{-\infty}^{\infty} |F(s)|^2 = 2\pi^2 \int_{0}^{\infty} e^{-4\pi s}\, ds$$

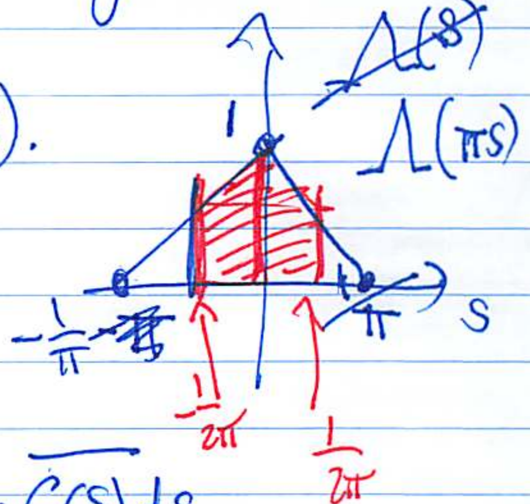
$$= \frac{2\pi^2}{-4\pi} e^{-4\pi s}\Big|_0^{\infty} = \frac{\pi}{2}.$$

④. (Parseval)

$$\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^3 dx$$

$$\frac{\sin x}{x} = \frac{\sin\left(\pi\cdot\frac{x}{\pi}\right)}{\pi\cdot x\cdot\frac{1}{\pi}} = \frac{\sin\left(\pi\frac{x}{\pi}\right)}{\pi\cdot\frac{x}{\pi}} = \operatorname{sinc}\left(\frac{x}{\pi}\right).$$

$$\left(\frac{\sin x}{x}\right)^3 = \frac{\sin x}{x}\cdot\left(\frac{\sin x}{x}\right)^2 = \underbrace{\operatorname{sinc}\left(\frac{x}{\pi}\right)}_{f(x)}\cdot\underbrace{\operatorname{sinc}^2\left(\frac{x}{\pi}\right)}_{g(x)}$$



$$F(s) = \frac{1}{\frac{1}{\pi}}\Pi\left(\frac{s}{\frac{1}{\pi}}\right) = \pi\cdot\Pi(\pi s).$$

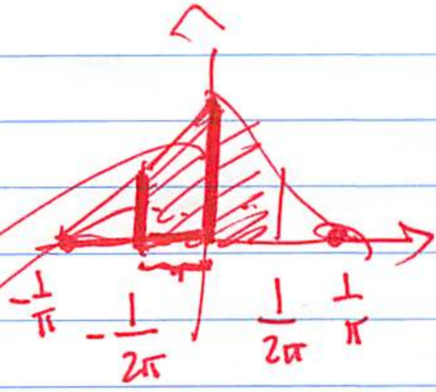
$$G(s) = \pi\Lambda(\pi s).$$

Parseval: $\displaystyle\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \int_{-\infty}^{\infty} F(s)\cdot\overline{G(s)}\,ds$

$$= \pi^2\int_{-\infty}^{\infty}\underbrace{\Pi(\pi s)}_{1\ :\ \left[-\frac{1}{2\pi},\ \frac{1}{2\pi}\right]}\Lambda(\pi s)\,ds = \pi^2\int_{-\frac{1}{2\pi}}^{\frac{1}{2\pi}}\Lambda(\pi s)\,ds$$

$$= \ldots = \frac{3}{4}\pi.$$

$$\pi^2 \int_{-\frac{1}{2\pi}}^{\frac{1}{2\pi}} \Lambda(\pi s)\, ds =$$

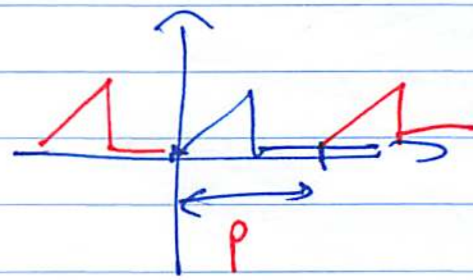


$$\pi^{\cancel{2}} \cdot \cancel{2} \cdot \frac{1}{\cancel{2}}\left(1+\frac{1}{2}\right)\cdot \frac{1}{2\cancel{\pi}} = \pi \cdot \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}\pi$$

Poisson's Relation

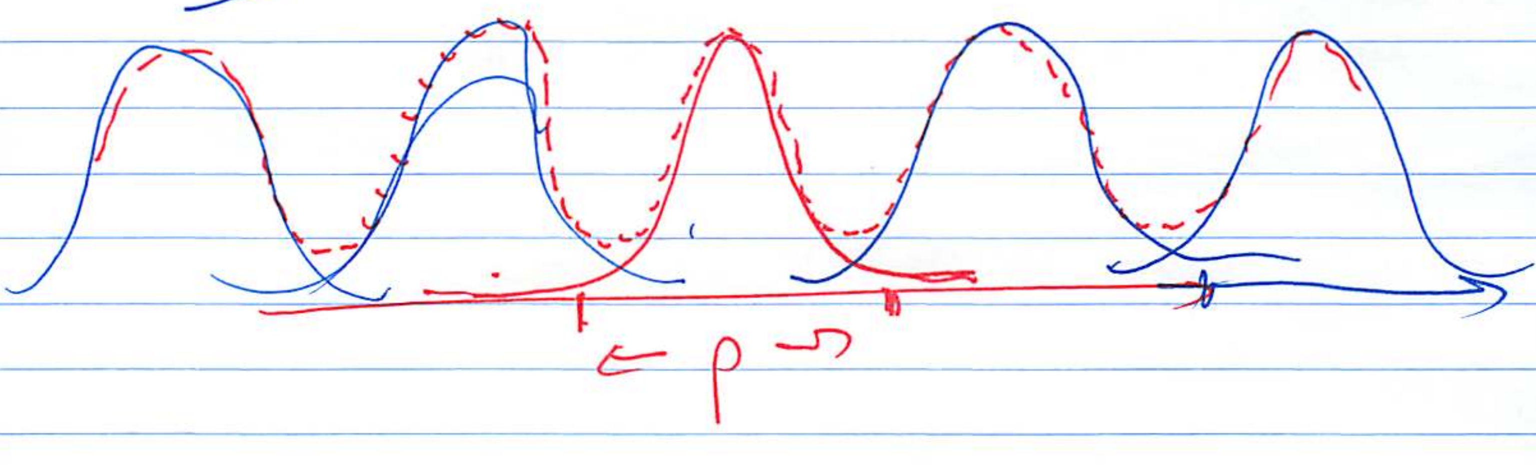
$f(x)$ $\mathbb{R}$.

Define a periodic function: p-periodic



$$g(x) = \sum_{m=-\infty}^{\infty} f(x - mp)$$

p-periodic.



$f(x)$ Known: $F(s)$

$g(x)$ ~~G(s)~~ $G[k] = ?$

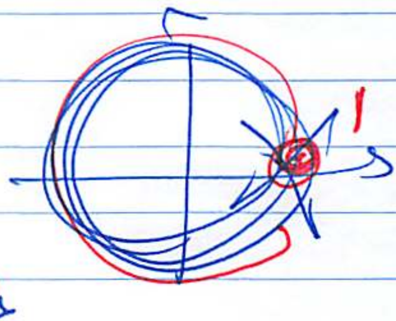
$$G[k] \overset{\text{def}}{=} \frac{1}{p}\int_0^p g(x)\, e^{-\frac{2\pi i k x}{p}}\, dx$$

F. Series

$$= \frac{1}{p}\int_0^p \sum_{m=-\infty}^{\infty} f(x-mp)\, e^{-\frac{2\pi i k x}{p}}\, dx$$

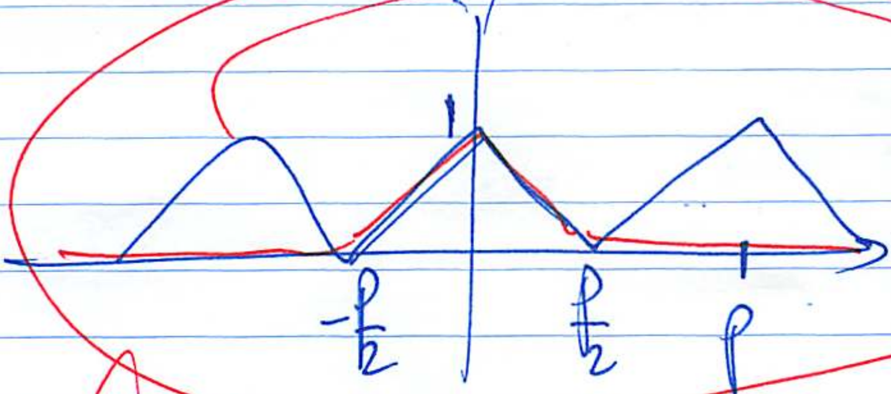
$$= \frac{1}{p}\sum_{m=-\infty}^{\infty}\int_0^p f(x-mp)\, e^{-\frac{2\pi i k (x-mp)}{p}}\, dx$$

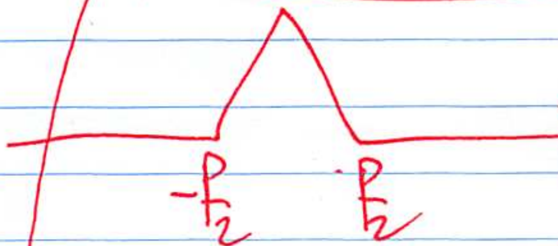
$$e^{+\frac{2\pi i k m p}{p}} = e^{2\pi i \overset{\mathbb{Z}}{k} \overset{\mathbb{Z}}{m}} = 1$$



$$= \frac{1}{p}\int_{-\infty}^{\infty} f(\xi)\, e^{-\frac{2\pi i k \xi}{p}}\, d\xi$$

$$\boxed{G[k] = \frac{1}{p} F\left(\frac{k}{p}\right).}$$





$$f(x) = \Lambda\left(\frac{x}{p/2}\right).$$

$$F(s) = \frac{p}{2}\operatorname{sinc}^2\left(\frac{ps}{2}\right)$$ Known

$$G[k] = \frac{1}{p}F\left(\frac{k}{p}\right) = \frac{1}{2}\operatorname{sinc}^2\left(\frac{k}{2}\right).$$

F. Series,

$$g(x) = \sum_{k=-\infty}^{\infty} G[k]\, e^{\frac{2\pi i k x}{p}}$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{2}\operatorname{sinc}^2\left(\frac{k}{2}\right) e^{\frac{2\pi i k x}{p}}.$$