CENTRE
DE RECHERCHES
MATHÉMATIQUES

Math 464 – Midterm #2 – Solutions

① ⓐ $f(x) = \delta(x) + \delta(x-1)$ $\qquad$ $g(x) = \delta(x+1) + \delta(x-1)$

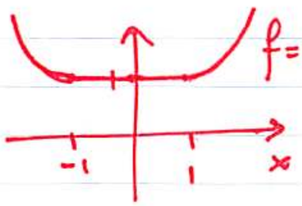
$\hat{f}(s) = 1 + e^{-2\pi i s}$ $\qquad$ $\hat{g}(s) = e^{2\pi i s} + e^{-2\pi i s}$.

ⓑ $\widehat{f*g}(s) = \hat{f}(s)\cdot\hat{g}(s) = (1+e^{-2\pi i s})(e^{2\pi i s} + e^{-2\pi i s})$

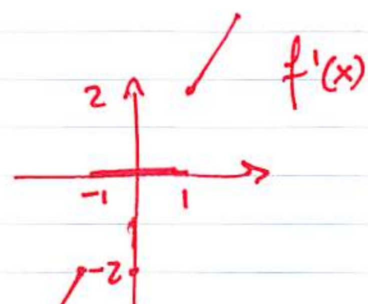
$= e^{2\pi i s} + 1 + e^{-2\pi i s} + e^{-4\pi i s}$

$\Rightarrow (f*g)(x) = \delta(x+1) + \delta(x) + \delta(x-1) + \delta(x-2)$.

② ⓐ



$$f(x) = \begin{cases} x^2 & x \le -1 \text{ or } x \ge 1 \\ 1 & -1 \le x \le 1 \end{cases}$$



$$f'(x) = \begin{cases} 2x & x < -1 \text{ or } x > 1 \\ 0 & -1 < x < 1 \end{cases}$$

$$f''(x) = 2\delta(x+1) + 2\delta(x-1) + \begin{cases} 2 & x<-1 \text{ or } x>1 \\ 0 & \text{otherwise} \end{cases}$$

$$f'''(x) = 2\delta'(x+1) + 2\delta'(x-1) - 2\delta(x+1) + 2\delta(x-1)$$

ⓑ $(2\pi i s)^3 F(s) = [2\cdot(2\pi i s)][e^{2\pi i s} + e^{-2\pi i s}] - 2[e^{2\pi i s} - e^{-2\pi i s}]$.

Since F is a regular function, we can divide by $(2\pi i s)^3$.

(3) (a) $$\int_{-\infty}^{\infty} \operatorname{sinc}(x)\, \text{Ш}(x)\, \phi(x)\, dx = \int_{-\infty}^{\infty} \text{Ш}(x) \left[\operatorname{sinc}(x)\, \phi(x) \right] dx = \qquad \forall \phi \in S$$

$$= \sum_{n=-\infty}^{\infty} \operatorname{sinc}(n)\, \phi(n) \underset{\Uparrow}{=} \phi(0) = \int_{-\infty}^{\infty} \delta(x)\, \phi(x)\, dx \implies \operatorname{sinc}(x)\, \text{Ш}(x) = \delta(x)$$

$$\operatorname{sinc}(n) = \frac{\sin(\pi n)}{\pi n} = \begin{cases} 0 & n \in \mathbb{N} \setminus \{0\} \\ 1 & n = 0 \end{cases}.$$

(b) $$\int_{-\infty}^{\infty} (\delta' * \delta')(x)\, e^{-\pi x^2} dx \underset{\Uparrow}{=} \int_{-\infty}^{\infty} \delta''(x)\, e^{-\pi x^2} dx = \int_{-\infty}^{\infty} \delta(x) \left(e^{-\pi x^2} \right)'' dx$$

$$\left(\widehat{\delta' * \delta'} = \widehat{\delta'} \cdot \widehat{\delta'} = (2\pi i s)^2 \implies \delta' * \delta' = \delta'' \right)$$

$$= \left(e^{-\pi x^2} \right)'' \Big|_{x=0} = -2\pi. \qquad \left(\left(e^{-\pi x^2} \right)'' = -2\pi e^{-\pi x^2} + (2\pi x)^2 e^{-\pi x^2} \right)$$

(4) Compute the Fourier Transform of 
and use Poisson's relation.