Bessel's inequality

Let f be a function on $\mathbb{T}_p$.

Let $\tau_n(x) = \sum_{k=-n}^{n} c_k e^{\frac{2\pi i k x}{p}}$ be any p-periodic trigonometric polynomial of degree n or less. (complex coeffs).

Then

$$\int_{x=0}^{p} |f(x) - \tau_n(x)|^2 dx = \int_{x=0}^{p} \left(f(x) - \sum_{k=-n}^{n} c_k e^{\frac{2\pi i k x}{p}}\right)\left(\overline{f(x)} - \sum_{\ell=-n}^{n} \overline{c_\ell} e^{-\frac{2\pi i \ell x}{p}}\right) dx$$

$$= \int_0^p |f(x)|^2 dx - \sum_{\ell=-n}^{n} \overline{c_\ell} \int_0^p f(x) e^{-\frac{2\pi i \ell x}{p}} dx - \sum_{k=-n}^{n} c_k \int_0^p \overline{f(x)} e^{\frac{2\pi i k x}{p}} dx$$

$$+ \sum_{k=-n}^{n} \sum_{\ell=-n}^{n} c_k \overline{c_\ell} \int_{x=0}^{p} e^{\frac{2\pi i k x}{p}} e^{-\frac{2\pi i \ell x}{p}} dx$$

$$= \int_0^p |f(x)|^2 dx - p \sum_{\ell=-n}^{n} \overline{c_\ell} F[\ell] - p \sum_{k=-n}^{n} c_k \overline{F[k]} + p \sum_{k=-n}^{n} c_k \overline{c_k}.$$

$$= \int_0^p |f(x)|^2 dx + p \sum_{k=-n}^{n} |F[k] - c_k|^2 - p \sum_{k=-n}^{n} |F[k]|^2.$$

(whenever the integrals exist & finite, e.g. when f is continuous at all but finitely many points of $\mathbb{T}_p$).

Set $c_k = F[k]$, $k \in \mathbb{Z}$.

$$\Rightarrow 0 \le \int_{x=0}^{p} \left| f(x) - \sum_{k=-n}^{n} F[k] e^{\frac{2\pi i k x}{p}} \right| = \int_0^p |f(x)|^2 dx - p \sum_{k=-n}^{n} |F[k]|^2 \quad \forall n \in \mathbb{N}.$$

$$\Rightarrow \forall n \in \mathbb{N} \quad \int_{x=0}^{p} |f(x)|^2 dx \ge p \sum_{k=-n}^{n} |F[k]|^2 \qquad \underline{\text{Bessel's inequality}}$$