Sketch of solutions to Final Exam - SPRING 2014

1. a. $f(x)$ is a convolution of a box function and a Gaussian. Its Fourier Transform is therefore the product of a sinc function and a Gaussian.

   b. The FT of the convolution is the product of the FTs of the functions. The FT of a Gaussian is a Gaussian so the FT of $f_0 * f_0$ is the product of Gaussians which is a Gaussian.
   Don't forget to convert back the Gaussian.

2. a. A straightforward calculation of the Fourier series of the given function.

   b. The value at $x=\frac{1}{2}$ should be the average of the left and right limits. $\left(=\frac{1}{4}\right)$.

3. Separation of variables $u(x,t) = X(x)T(t)$.

$$\Rightarrow \frac{T'}{9T} = \frac{X''}{X} = \text{const}.$$

First consider $\frac{X''}{X} = \text{const}$ with the boundary conditions:

$$X'(0) = X'(1) = 0.$$

The const can be zero, positive, or negative. All 3 cases should be checked. Solutions will exist for

$$X'' = 0 \quad \text{and} \quad X'' = -\lambda^2 X.$$

The solutions end up being $X_n(x) = \cos(\lambda_n x)$ with $\lambda_n = n\pi$
$n = 0, 1, \ldots$

(The case $n=0$ takes care also of $X'' = 0$).

For the time-dependent part: $T' = -\lambda_n^2 9 T$

$$\Rightarrow T(t) = T(0)\, e^{-9\lambda_n^2 t} \quad (\lambda_n = n\pi).$$

Hence the general solution of the PDE + the boundary conditions is:

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos(n\pi x)\, e^{-9n^2\pi^2 t}.$$

This still has to satisfy the initial conditions: $u(x,0) = \cos(4\pi x)$ which implies that $a_4 = 1$ while $a_n = 0$ for $n \neq 4$.
(If the initial data was more complex, an expansion of the initial data as into Fourier series would be an extra step).

$\Rightarrow$ The solution is $u(x,t) = \cos(4\pi x)\, e^{-9\cdot 16\pi^2 t}$.

4. (a) + (b) Straightforward from the properties of the Fourier transform of generalized functions.