The Fourier Transform of a Gaussian

$f(x) = e^{-\pi x^2}$.

The Fourier Transform of $f(x)$ is:

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x}\, dx = \int_{-\infty}^{\infty} e^{-\pi x^2} \cos(2\pi s x)\, dx$$

(integration by parts)

$$= \cancel{\frac{e^{-\pi x^2} \sin(2\pi s x)}{2\pi s} \Bigg|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} \frac{e^{-\pi x^2}(-2\pi x)\sin(2\pi s x)}{2\pi s}\, dx$$

Now:

$$F'(s) = \int_{-\infty}^{\infty} e^{-\pi x^2} \frac{\partial}{\partial s} \cos(2\pi s x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\pi x^2} (-2\pi x) \sin(2\pi s x)\, dx.$$

Hence $\quad F'(s) + 2\pi s F(s) = 0.$

$$\Rightarrow \quad \left(F(s) e^{\pi s^2}\right)' = F'(s) e^{\pi s^2} + F(s)\, 2\pi s\, e^{\pi s^2}$$

$$= e^{\pi s^2}\left(F' + 2\pi s F\right) = 0.$$

$$\Rightarrow \quad F(s) e^{\pi s^2} = \text{const} = F(0)$$

$$\Rightarrow \quad \boxed{F(s) = F(0) e^{-\pi s^2}}$$

It remains to show that $F(0) = 1$.