Fourier Transform for generalized functions

f	$F\tau$	
$c_1 f_1(x) + c_2 f_2(x)$	$c_1 f_1^\wedge(s) + c_2 f_2^\wedge(s)$.	$c_1, c_2 \in \mathbb{C}$.
$f(-x)$	$f^\wedge(-s)$	
$f^-(x)$	$f^{\wedge -}(-s)$	
$f(x-x_0)$	$e^{-2\pi i s x_0} f^\wedge(s)$	
$e^{2\pi i s_0 x} f(x)$	$f^\wedge(s-s_0)$	
$f^\wedge(x)$	$f(-s)$	
$f(ax)$	$\frac{1}{\lvert a\rvert} f^\wedge(\frac{s}{a})$	$a>0$ or $a<0$.
$f^{(n)}(x)$	$(2\pi i s)^n f^\wedge(s)$	$n=1,2,\dots$
$x^n f(x)$	$(-2\pi i)^{-n} [f^\wedge]^{(n)}(s)$.	$n=1,2,\dots$
$\alpha(x) f(x)$	$(\alpha^\wedge * f^\wedge)(s)$	$\alpha, \alpha', \dots$ CSG
$(\beta * f)(x)$	$\beta^\wedge(s) \cdot f^\wedge(s)$	$\beta^\wedge, \beta^{\wedge\prime}, \dots$ CSG

Recall the definition: $f^\wedge\{\phi\} = f\{\phi^\wedge\}$.

Examples:

$\delta^\wedge(s) = 1$. (We already showed that).

$$\begin{cases} (\delta^{(n)})^\wedge = (2\pi i s)^n \\ (\delta(x-x_0))^\wedge = e^{-2\pi i s x_0} \end{cases}$$

$$\begin{cases} (x^n)^\wedge = (-2\pi i)^{-n} \delta^{(n)}(s) \\ (e^{2\pi i s_0 x})^\wedge = \delta(s-s_0). \end{cases}$$

[by inverting the first 2 formulas]

Example:

$$f(x) = x^3 - 3x^2 + 3x - 1.$$

$$F(s) = \frac{\delta'''(s)}{(-2\pi i)^3} - \frac{3\delta''(s)}{(-2\pi i)^2} + \frac{3\delta'(s)}{-2\pi i} - \delta(s).$$

But also $f(x) = (x-1)^3$
which means that

$$F(s) = e^{-2\pi i s} \cdot \frac{\delta'''(s)}{(-2\pi i)^3}.$$

And these expressions are identical.
To see that:

$$\alpha(s)\,\delta^{(k)}(s-s_0) = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \alpha^{(k-l)}(s_0)\,\delta^{(l)}(s-s_0).$$

Hence
$$e^{-2\pi i s} \cdot \delta'''(s) = \binom{3}{0}(-1)^{3-0}(-2\pi i)^3 \delta(s)$$
$$+ \binom{3}{1}(-1)^{3-1}(-2\pi i)^2 \delta'(s)$$
$$+ \dots$$

Example: $f(x) = \cos(2\pi x) = \dfrac{e^{2\pi i x} + e^{-2\pi i x}}{2}$

$$\Rightarrow F(s) = \frac{1}{2}\left[\delta(s-1) + \delta(s+1)\right].$$

$$f(x) = \sin(2\pi x) = \frac{e^{2\pi i x} - e^{-2\pi i x}}{2i}$$

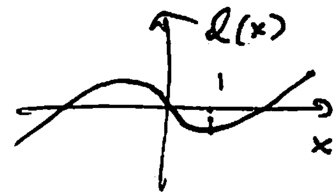
$$\Rightarrow F(s) = \frac{1}{2i}\left[\delta(s-1) - \delta(s+1)\right].$$

The inverse power function

Goal: Find the FT of expressions such as $\frac{1}{x}$. For this we have to start by providing a framework for considering $\frac{1}{x}$ as a generalized function.

The functions $\frac{1}{x}, \frac{1}{x^2}, \dots$

Let $\ell(x) = \int_0^x \log|u|\,du = x\log|x| - x$.



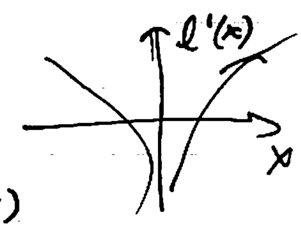
This is a CSG function.

$$\ell\{\phi\} = \int_{-\infty}^{\infty} (x\log|x| - x)\,\phi(x)\,dx \qquad \phi \in S.$$

$$\Rightarrow \ell'\{\phi\} = -\int_{-\infty}^{\infty} \ell(x)\phi'(x)\,dx = -\int_{-\infty}^{\infty} (x\log|x| - x)\,\phi'(x)\,dx$$

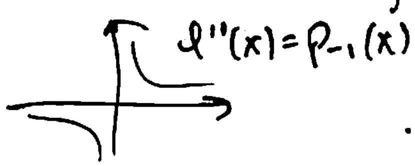
$$= -\left[(x\log|x| - x)\,\phi\right]\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \phi(x)\left[\frac{x}{|x|} + \log|x| - 1\right]dx$$

$$= \int_{-\infty}^{\infty} \log|x|\,\phi(x)\,dx \quad \Rightarrow \quad \ell'(x) = \log|x|.$$



For ordinary functions $\frac{1}{x} = \ell''(x)$, $x^{-2} = -\ell'''(x)$, $x^{-3} = \ell''''(x)/2!, \dots$

Define the inverse power functions: $p_{-1} = \ell''$, $p_{-2} = -\ell'''$, $p_{-3} = \ell^{(4)}/2!, \dots$



This is a little tricky since $\displaystyle\int_{-\infty}^{\infty}\frac{\phi(x)}{x^n}\,dx$

will not be defined unless ϕ has a zero of multiplicity n at 0.

The work around is by defining

$$P_{-1}\{\phi\} = \lim_{L\to\infty}\int_{-L}^{L}\frac{\phi(x)-\phi(0)}{x}\,dx = \int_{0}^{\infty}\frac{\phi(x)-\phi(-x)}{x}\,dx$$

$$P_{-2}\{\phi\} = \lim_{L\to\infty}\int_{-L}^{L}\frac{\phi(x)-\phi(0)-x\phi'(0)}{x^2}\,dx$$

$$\int_{-L}^{0}\frac{\phi(x)-\phi(0)}{x}\,dx = -\int_{0}^{L}\frac{\phi(-x)-\phi(0)}{x}\,dx$$

etc.

$$P_{n}\{\phi\} = \lim_{L\to\infty}\int_{-L}^{L}x^{-n}\left\{\phi(x)-\sum_{k=0}^{n-1}\frac{1}{k!}x^{k}\phi^{(k)}(0)\right\}dx$$

<u>The FT of P_{-1}:</u>

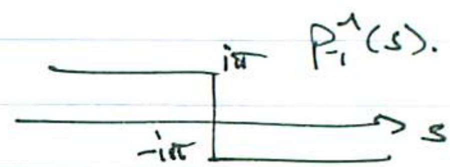
$$P_{-1}^{\wedge}\{\phi\} = \int_{-\infty}^{\infty}P_{-1}^{\wedge}(s)\phi(s)\,ds \overset{\text{Parseval}}{=} \int_{-\infty}^{\infty}P_{-1}(x)\phi^{\wedge}(x)\,dx$$

$$= \int_{0}^{\infty}\frac{\phi^{\wedge}(x)-\phi^{\wedge}(-x)}{x}\,dx = \int_{x=0}^{\infty}\int_{s=-\infty}^{\infty}\phi(s)\left(\frac{e^{-2\pi i s x}-e^{2\pi i s x}}{x}\right)ds\,dx$$

(p.395)

$$= \dots = \int_{-\infty}^{\infty}\left(-\pi i\,\mathrm{sgn}(s)\right)\phi(s)\,ds$$

i.e. $P_{-1}^{\wedge}(s) = -\pi i\,\mathrm{sgn}(s)$



Given that $P_{-1}^{\wedge}(s) = -\pi i \, \mathrm{sgn}(s)$.

We can write $P_{-n}(x) = \dfrac{(-1)^{n-1}}{(n-1)!} P_{-1}^{(n-1)}(x) \quad n = 1, 2, \ldots$

and thus

$$P_{-n}^{\wedge}(s) = -\pi i \, \mathrm{sgn}(s) \, (2\pi i s)^{n-1} \frac{(-1)^{n-1}}{(n-1)!}$$

$$= -\pi i \, \mathrm{sgn}(s) \frac{(-2\pi i s)^{n-1}}{(n-1)!}.$$

<u>Example:</u> Find the FT of $f(x) = \dfrac{x^2+1}{x^2-1}$.

$$f(x) = \frac{x^2+1}{x^2-1} = 1 + \frac{1}{x-1} + \frac{1}{x+1} = 1 + P_{-1}(x-1) + P_{-1}(x+1).$$

$$\Rightarrow F(s) = \delta(s) + e^{-2\pi i s}(-\pi i)\mathrm{sgn}(s) - e^{+2\pi i s}(-\pi i)\mathrm{sgn}\, s$$

$$= \delta(s) + \pi i \, \mathrm{sgn}\, s \left[\frac{e^{2\pi i s} - e^{-2\pi i s}}{2i} \right] \cdot (2i)$$

$$= \delta(s) \; 2\pi \, \mathrm{sgn}\, s \cdot \sin(2\pi s)$$

$$= \delta(s) - 2\pi \sin(2\pi |s|).$$

$$\alpha(x)\delta^{(k)}(x-x_0) = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \alpha^{(k-l)}(x_0)\delta^{(l)}(x-x_0).$$

Consider $(x-x_0)^n f(x) = 0$. ⊛

$\alpha(x) = (x-x_0)^n$ & its derivatives are CSG.
But $\frac{1}{\alpha}$ has a singularity at $x = x_0$.

$(x-x_0)^n \delta^{(k)}(x-x_0) = 0$ when $k = 0, \dots, n-1$.

Hence, the generalized function
$f(x) = c_0\delta(x-x_0) + c_1\delta'(x-x_0) + \dots + c_{n-1}\delta^{(n-1)}(x-x_0)$
satisfies ⊛ for any choice of constants $c_0, \dots, c_{n-1}$.

In fact, any generalized function satisfying ⊛ must have this form.
To prove that: $(x-x_0)^n f(x) = 0$.
Let $g(y) = f(y+x_0)$ $\quad y = x-x_0 \Rightarrow x = y+x_0$.

$$y^n g(y) = 0 \Rightarrow (-2\pi i)^{-n}[g^\wedge]^{(n)}(s) = 0$$
$$\Rightarrow [g^\wedge]^{(n)}(s) = 0$$

$$\Rightarrow [g^\wedge]^{(n-1)}(s) = \tilde{c}_{n-1}$$
$$[g^\wedge]^{(n-2)}(s) = \tilde{c}_{n-1}s + \tilde{c}_{n-2}$$
$$\vdots$$
$$[g^\wedge](s) = \tilde{c}_{n-1}s^{n-1} + \dots + \tilde{c}_0$$
$$\Rightarrow g(y) = c_0\delta(y) + \dots + c_{n-1}\delta^{(n-1)}(y)$$
$$\Rightarrow f(x) = c_0\delta(x-x_0) + \dots + c_{n-1}\delta^{(n-1)}(x-x_0).$$

Examples:

(1) $$x\cdot[f(x)-P_{-1}(x)]=0 \quad \Rightarrow \quad f(x)=P_{-1}(x)+c\delta(x).$$

(2) $$x^2 f(x)=\sin^2(\pi x)$$

$$\Rightarrow \quad x^2\left(f(x)-\pi^2 \operatorname{sinc}^2(x)\right)=0$$

$$\Rightarrow \quad f(x)=\pi^2\operatorname{sinc}^2(x)+c_0\delta(x)+c_2\delta'(x).$$

(3) $$(x^2-1)f(x)=1. \quad \Rightarrow \text{formally} \quad (x^2-1)\left(f(x)-\frac{1}{x^2-1}\right)=0.$$

$$\frac{1}{x^2-1}=\frac{1}{2}\left[\frac{1}{x-1}-\frac{1}{x+1}\right]=\frac{1}{2}\left[P_{-1}(x-1)-P_{-1}(x+1)\right].$$

$$\Rightarrow (x^2-1)\left[f(x)-\frac{1}{2}P_{-1}(x-1)+\frac{1}{2}P_{-1}(x+1)\right]=0$$

$$\Rightarrow f(x)=\frac{1}{2}P_{-1}(x-1)-\frac{1}{2}P_{-1}(x+1)+c\delta(x-1)+d\delta(x+1).$$

Here, we need to use the result from 7.24.

$$(x^2-1)\cdot\frac{1}{2}\left[P_{-1}(x-1)-P_{-1}(x+1)\right].$$

$$=\frac{1}{2}\left[(x+1)(x-1)P_{-1}(x-1)-(x-1)(x+1)P_{-1}(x+1)\right]$$

$$=\frac{1}{2}\left[(x+1)\cdot 1-(x-1)\cdot 1\right]=1.$$

Using $x\cdot P_{-1}(x)=1$. Since

$$\int_{-\infty}^{\infty} x\,P_{-1}(x)\phi(x)\,dx=\int_{-\infty}^{\infty}P_{-1}(x)\left[\phi(x)\cdot x\right]=\int_0^{\infty}\frac{x\phi(x)-(-x)\phi(-x)}{x}dx$$

$$=\int_{-\infty}^{\infty}1\cdot\phi(x)\,dx.$$

Example: (7.24).

Let Polynomial P have real roots $\alpha_1 < \dots < \alpha_r$ with multiplicities $n_1, \dots, n_r$.

[The other roots (if any) have nonzero imaginary parts]

Let f be a g. function.

Assume that $P(x) \cdot f(x) = 0$.

Show that f can be written as

$$f(x) = \sum_{\mu=1}^{r} \sum_{\nu=0}^{n_\mu - 1} c_{\mu\nu}\, \delta^{(\nu)}(x - \alpha_\mu)$$

for some const. $c_{\mu\nu}$.

Solution: $\cancel{\{(x-\alpha_1)^{n_1} \cdots (x-\alpha_r)^{n_r}\} \cdot f(x) = a \cdot h}$

Examples:

① Solve the DE $y^{(4)}(x) - y(x) = 0$.

Solution: $\left[(2\pi i s)^4 - 1\right] Y(s) = 0$.

$$\left[(2\pi i s)^2 + 1\right]\left[2\pi i s + 1\right]\left[2\pi i s - 1\right] Y(s) = 0.$$

$$\left(s + \tfrac{1}{2\pi}\right)\left(s - \tfrac{1}{2\pi}\right) \cdot Y(s) = 0.$$

$$\Rightarrow Y(s) = c\,\delta\left(s - \tfrac{1}{2\pi}\right) + d\,\delta\left(s + \tfrac{1}{2\pi}\right)$$

$$\Rightarrow y(x) = c e^{ix} + d e^{-ix}.$$

②. Solve $y''(t) + \omega^2 y(t) = \frac{F}{m}\delta(t)$

with a solution that vanishes for $t<0$.

Solution: $\left[(2\pi i s)^2 + \omega^2\right] Y(s) = \frac{F}{m}$

$$\left(s - \tfrac{\omega}{2\pi}\right)\left(s + \tfrac{\omega}{2\pi}\right) Y(s) = -\frac{F}{4\pi^2 m}.$$

$$\Rightarrow Y(s) = c\,\delta\left(s - \tfrac{\omega}{2\pi}\right) + d\,\delta\left(s + \tfrac{\omega}{2\pi}\right) - \frac{F}{4\pi^2 m}\left(\frac{1}{s - \frac{\omega}{2\pi}} - \frac{1}{s + \frac{\omega}{2\pi}}\right).$$

$$y(t) = c e^{i\omega t} + d e^{-i\omega t} - \frac{F}{4\pi m\omega}\left(e^{i\omega t} - e^{-i\omega t}\right)(i\pi)\,\mathrm{sgn}(t).$$

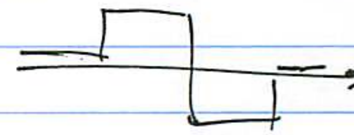
$$= c e^{i\omega t} + d e^{-i\omega t} + \frac{F}{2m\omega}\sin(\omega t)\,\mathrm{sgn}(t)$$

For y to vanish at $t<0$ we need $c e^{i\omega t} + d e^{-i\omega t} = \frac{F}{2m\omega}\sin(\omega t)$

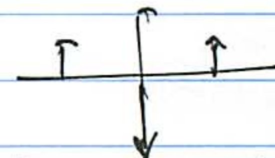
$$\Rightarrow \boxed{y(t) = \frac{F}{m\omega}\sin(\omega t)\,h(t)}$$

Example: Find the FT of $f(x) = \Lambda(x)$.

$$f'(x) = \begin{cases} 0 & x<-1 \\ 1 & -1<x<0 \\ -1 & 0<x<1 \\ 0 & 1<x \end{cases}$$



$$f''(x) = \delta(x+1) - 2\delta(x) + \delta(x-1).$$



FT: $(2\pi i s)^2 F(s) = e^{2\pi i s}\cdot 1 - 2\cdot 1 + e^{-2\pi i s}\cdot 1 = (e^{\pi i s} - e^{-\pi i s})^2$

$$\Rightarrow \quad s^2\left[F(s) - \text{sinc}^2(s)\right] = 0.$$

$$\Rightarrow \quad F(s) = \text{sinc}^2(s) + c_0\delta(s) + c_1\delta'(s).$$

But $F(s)$ should be an ordinary function which means that $c_0 = c_1 = 0$.

Example: Find the FT of $f(x) = \frac{1}{2}\text{sgn}(x)$ & $h(x) = \begin{cases} 1 & x>0 \\ 0 & x<0. \end{cases}$

$$f'(x) = h'(x) = \delta(x).$$

$$\Rightarrow \quad 2\pi i s\cdot F(s) = 2\pi i s\cdot H(s) = 1.$$

$$\Rightarrow \quad s\cdot\left[F(s) - \frac{P_{-1}(s)}{2\pi i}\right] = s\cdot\left[H(s) - \frac{P_{-1}(s)}{2\pi i}\right] = 0.$$

$$\Rightarrow \quad F(s) = \frac{P_{-1}(s)}{2\pi i} + c\delta(s), \quad H(s) = \frac{P_{-1}(s)}{2\pi i} + d\delta(s).$$

$f(x)$ is odd $\Rightarrow$ $F(s) = $ odd $\Rightarrow$ $c=0$ since δ is even.

$$\Rightarrow \quad F(s) = \frac{1}{2\pi i s}.$$

Now $h(x) = f(x) + \frac{1}{2}$ $\Rightarrow$ $H(s) = \frac{1}{2\pi i s} + \frac{1}{2}\delta(s).$