MATH 464 Homework 8

7.14 (a) $\beta(x)=\delta(ax+b)$, $f(x)=\cos(cx+d)$

$$\hat{\beta}(s)=\frac{1}{a}\left(\delta\left(x+\frac{b}{a}\right)\right)^\wedge=\frac{1}{a}e^{2\pi i\frac{b}{a}s}$$

$$(\beta*f)^\wedge=\hat{\beta}\cdot\hat{f}=\frac{1}{a}e^{2\pi i\frac{b}{a}s}\cdot\hat{f}\overset{\mathcal{F}^{-1}}{\Longrightarrow}\frac{1}{a}f\left(x+\frac{b}{a}\right)=\frac{1}{a}\cos\left(cx+\frac{bc}{a}+d\right)$$

(b) $\beta(x)=e^{-ax^2}$, $f(x)=\cos(cx+d)$

$$\hat{\beta}(s)=\sqrt{\frac{\pi}{a}}e^{-\frac{\pi^2}{a}s^2},$$

$$\hat{f}(s)=\cos\left(c\left(x+\frac{d}{c}\right)\right)=e^{2\pi i\frac{d}{c}s}(\cos(cx))^\wedge$$

$$=e^{2\pi i\frac{d}{c}s}\left[\frac{1}{2}\delta\left(s+\frac{c}{2\pi}\right)+\frac{1}{2}\delta\left(s-\frac{c}{2\pi}\right)\right]$$

$$=\frac{1}{2}e^{2\pi i\frac{d}{c}s}\left[\delta\left(s+\frac{c}{2\pi}\right)+\delta\left(s-\frac{c}{2\pi}\right)\right]$$

$$\hat{\beta}\,\hat{f}=\frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-\frac{\pi^2}{a^2}s^2}e^{2\pi i\frac{d}{c}s}\left[\delta\left(s+\frac{c}{2\pi}\right)+\delta\left(s-\frac{c}{2\pi}\right)\right]$$

$$\beta*f=\frac{1}{2}\sqrt{\frac{\pi}{a}}\int_{-\infty}^{\infty}e^{-\frac{\pi^2}{a^2}s^2}e^{2\pi i\frac{d}{c}s}\left[\delta\left(s+\frac{c}{2\pi}\right)+\delta\left(s-\frac{c}{2\pi}\right)\right]e^{2\pi isx}ds$$

$$=\frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-\frac{\pi^2}{a^2}\left(\frac{c}{2\pi}\right)^2}\left[e^{2\pi i\left(\frac{d}{c}+x\right)\left(\frac{c}{2\pi}\right)}+e^{-2\pi i\left(\frac{d}{c}+x\right)\left(\frac{c}{2\pi}\right)}\right]$$

$$=\frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-\frac{\pi^2}{a^2}\frac{c^2}{4\pi^2}}\,2\cos[cx+d]$$

$$=\sqrt{\frac{\pi}{a}}\,e^{-\frac{c^2}{4a^2}}\cos[cx+d]$$

(c) $\beta(x) = \Pi(x)$, $f(x) = Ш^{(n)}(x)$

$\hat{\beta}(s) = \text{sinc}(s)$, $\hat{f}(s) = (2\pi i s)^n \cdot Ш(s)$

$\hat{\beta}(s) \cdot \hat{f}(s) = (2\pi i s)^n \,\text{sinc}(s)\, Ш(s) = 0$ for all s.

$\beta * f = 0$.

(d) $\beta(x) = \delta^{(n)}(x)$, $f(x) = \delta^{(m)}(x)$

$\hat{\beta}(s) = (2\pi i s)^n \quad \hat{f}(s) = (2\pi i s)^m$

$\hat{\beta} \cdot \hat{f} = (2\pi i s)^{m+n}$

$\beta * f = \delta^{(m+n)}(x)$.

(e) $\beta(x) = \delta(ax+b)$, $f(x) = Ш(x)$

$\hat{\beta}(s) = \frac{1}{a} e^{2\pi i \frac{b}{a} s} \quad \hat{f}(s) = Ш(s)$

$\hat{\beta} \cdot \hat{f} = \frac{1}{a} e^{2\pi i \frac{b}{a} s} Ш(s)$

$\beta * f = \frac{1}{a} Ш\left(x + \frac{b}{a}\right)$

(f) $\beta(x) = \delta'(ax+b)$, $f(x) = \text{sgn}(x)$

$\hat{\beta}(s) = e^{2\pi i s \frac{b}{a}} \left(\delta'(ax)\right)^{\wedge} = e^{2\pi i s \frac{b}{a}} \frac{1}{|a|} 2\pi i \left(\frac{s}{a}\right)$

$\hat{f}(s) = \frac{1}{\pi i s}$

$\hat{\beta}(s) \cdot \hat{f}(s) = e^{2\pi i s \frac{b}{a}} \frac{2}{a|a|}$

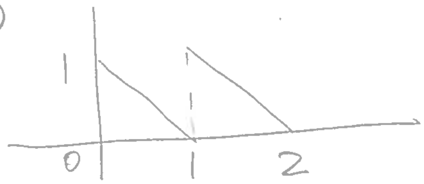
$\beta * f = \frac{2}{a} \delta(ax+b)$

Alternatively,

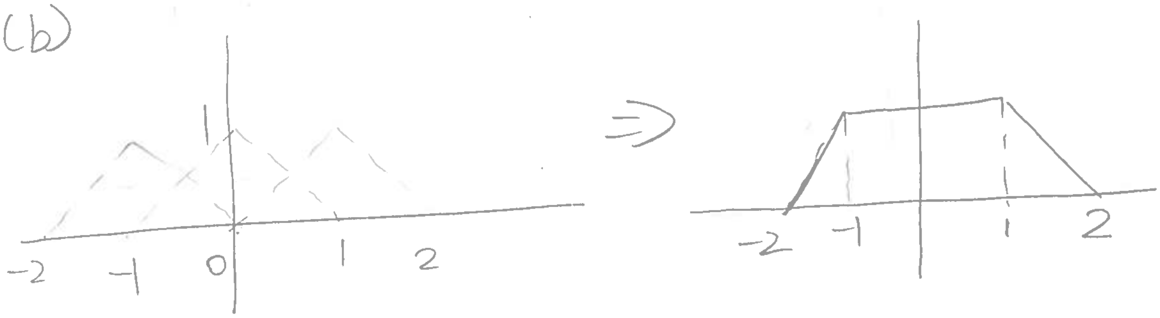
$\delta'(ax+b) * \text{sgn}(x)$
$= \delta(ax+b) * (\text{sgn}(x))'$
$= \delta(ax+b) * 2$
$= \frac{2}{a} \delta(ax+b)$

7.15

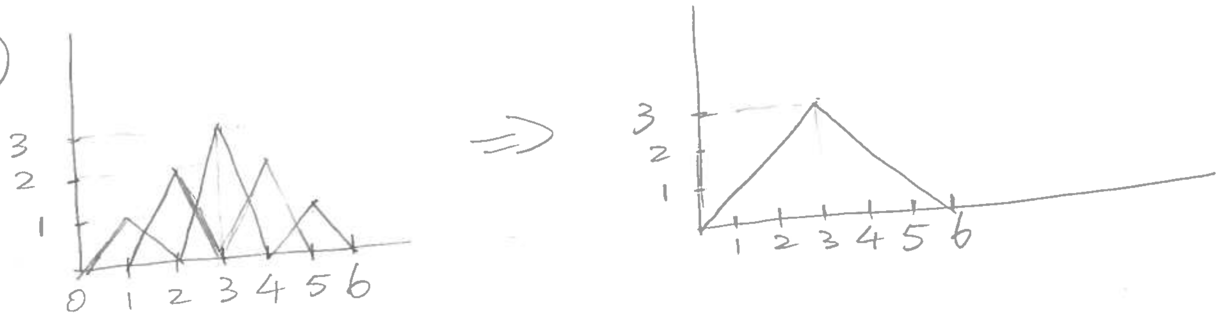
(a)


1
0
1
2

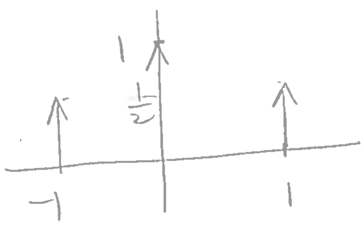
(b)


1
-2
-1
0
1
2
⇒
-2
-1
1
2

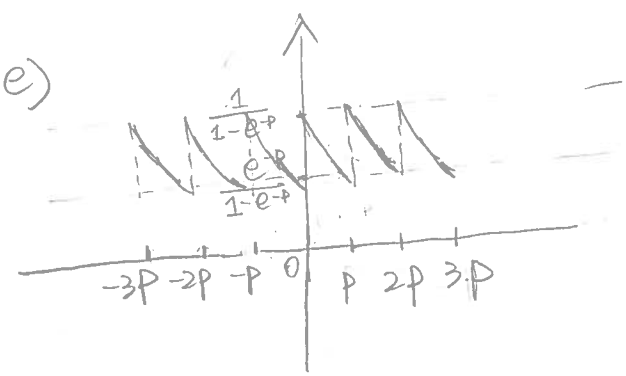
(c)


3
2
1
0 1 2 3 4 5 6
⇒
3
2
1
1 2 3 4 5 6

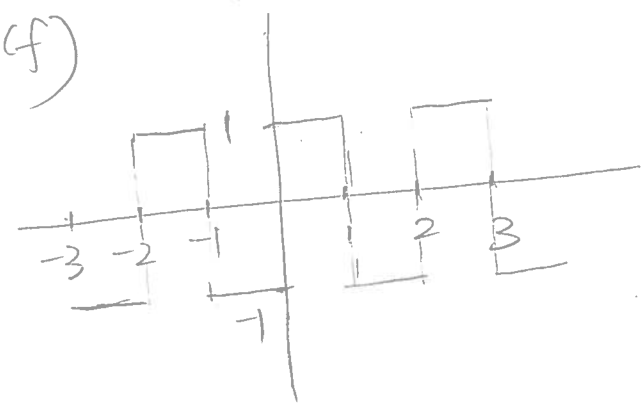
(d)


1
1/2
-1
1

(e)


1/(1-e^-p)
e^-p/(1-e^-p)
-3p -2p -p 0 p 2p 3p

(f)


1
-3
-2
-1
2
3
-1

7.33

(a) $f(x) = \delta(x-1) + \delta(x+1)$

$$F(s) = e^{-2\pi i s} + e^{2\pi i s} = 2\cos(2\pi s)$$

(c) $f(x) = \delta'(5x)$

$$F(s) = (2\pi i s)\cdot \frac{1}{5} = \frac{2}{5}\pi i s$$

(e) $f(x) = \sin(4\pi x)$

$$F(s) = \frac{i}{2}\left[\delta(s+2) - \delta(s-2)\right]$$

(g) $f(x) = \delta'(x) * x$

$$F(s) = (2\pi i s)\left(\frac{1}{-2\pi i}\right)\delta'(s)$$

$$= -s\,\delta'(s)$$

(i) $f(x) = 2\pi x \cdot \cos(2\pi x)$

$$F(s) = \mathcal{F}[2\pi x] * \mathcal{F}[\cos(2\pi x)]$$

$$= 2\pi \cdot (-2\pi i)^{-1}\delta^{(1)}(s) * \frac{1}{2}\left[\delta(s+1) + \delta(s-1)\right]$$

$$= \frac{i}{2}\left[\delta^{(1)}(s+1) + \delta^{(1)}(s-1)\right]$$

(k) $f(x) = e^{-\pi x^2}\cdot 2\pi x$

$$F(s) = (e^{-\pi s^2}) * 2\pi(-2\pi i)^{-1}\delta^{(1)}(s) = i(e^{-\pi s^2})' = -2\pi s i e^{-\pi s^2}$$

(m) $f(x) = \delta(2x+1) * \delta(4x+2)$

$$F(s) = \mathcal{F}[\delta(2x+1)]\cdot\mathcal{F}[\delta(4x+2)]$$

$$= \frac{1}{2}e^{\pi i s}\cdot\frac{1}{4}\cdot e^{\pi i s} = \frac{1}{8}e^{2\pi i s}$$

(5)

(e) $f(x) = \cos(x) \cdot \delta''(x)$

$$F(s) = \mathcal{F}[\cos(x)] * \mathcal{F}[\delta''(x)]$$

$$= \frac{1}{2}\left[\delta(s-\tfrac{1}{2\pi}) + \delta(s+\tfrac{1}{2\pi})\right] * (2\pi i s)^2$$

$$= -2\pi^2\left[(s-\tfrac{1}{2\pi})^2 + (s+\tfrac{1}{2\pi})^2\right] = -4\pi^2 s^2 - 1$$

7.40

(a) $f''(x) = \delta(x-1) - \delta(x+1) + 2\delta'(x)$

$$(f''(x))^\wedge = e^{-2\pi i s} - e^{2\pi i s} - 2\cdot(2\pi i s)$$

On the other hand,

$$(2\pi i s)^2 (f(x))^\wedge = (f''(x))^\wedge = -2i\sin(2\pi s) - 4\pi i s$$

$$\Rightarrow F(s) = -\frac{1}{4\pi^2 s^2}\left[-2i\sin(2\pi s) - 4\pi i s\right]$$

$$= \frac{i}{2\pi^2 s^2}\sin(2\pi s) + \frac{i}{\pi s} + C_0\delta(s) + C_1\delta'(s)$$

(c) $f'(x) = 2\delta(x) - 3\delta(x-1) + 2\delta(x-2)$

$$(2\pi i s)F(s) = 2 - 3e^{-2\pi i s} + 2e^{-4\pi i s}$$

$$F(s) = \frac{1}{2\pi i s}\left[2 - 3e^{-2\pi i s} + 2e^{-4\pi i s}\right] + C_0\delta(s)$$

(b) $f''(x) = \frac{1}{2}\delta(x) - \frac{3}{2}\delta(x-2) + \delta(x-3)$

$(2\pi i s)^2 F(s) = \frac{1}{2} - \frac{3}{2}e^{-4\pi i s} + e^{-6\pi i s}$

$\Rightarrow F(s) = \frac{1}{-4\pi s^2}\left(\frac{1}{2} - \frac{3}{2}e^{-4\pi i s} + e^{-6\pi i s}\right) + C_0\delta(s) + C_1\delta'(s)$

(d) $f'(x) = 2\delta(x+1) - 2\delta(x) + 2\delta(x-1)$

$(2\pi i s)F(s) = 2e^{2\pi i s} - 2 + 2e^{-2\pi i s}$

$\Rightarrow F(s) = \frac{1}{\pi i s}\left(e^{2\pi i s} + e^{-2\pi i s} - 1\right)$

$\hookrightarrow$ No $\delta(s)$ terms since $f_m = -f_0$